Adaptive high-order FEM for the mixed $\{U,p\}$ poro-elastic equations

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Motivation

- Non expert users
- Commercial software – large applications
- Adjust the order automatically based on
  - Mesh characteristics
  - Frequency
  - Type of materials
Outline of the presentation

1. Biot formulation and $p$-FEM numerical model
2. Literature review and objectives
3. Fixed-order cost analysis
4. Adaptive order strategy
5. First results
6. Conclusions
1. Biot formulation and $p$-FEM numerical model
Weak form: Mixed \{u,p\} formulation

\[ \int_{\Omega} \sigma^s(u) : \varepsilon^s(u^*) \, d\Omega - \omega^2 \int_{\Omega} \tilde{\rho} \, \mathbf{u} \cdot \mathbf{u}^* \, d\Omega \]

\[ + \int_{\Omega} \left[ \frac{\phi^2}{\omega^2} \nabla p^* \cdot \nabla p - \frac{\phi^2}{R} p^* p \right] \, d\Omega \]

\[ - \int_{\Omega} (\gamma + \phi') \left( \nabla p^* \cdot \mathbf{u} + \mathbf{u}^* \cdot \nabla p \right) \, d\Omega - \int_{\Omega} \phi' \left( p^* \nabla \cdot \mathbf{u} + p \nabla \cdot \mathbf{u}^* \right) \, d\Omega \]

\[ - \int_{\Gamma} \left( \sigma^t \cdot \mathbf{n} \right) \cdot \mathbf{u}^* \, d\Gamma - \int_{\Gamma} p^* \phi (\mathbf{U} - \mathbf{u}) \cdot \mathbf{n} \, d\Gamma = 0. \]

- 4 unknowns in 3D (skeleton displacement and pressure)
- Direct Solver (MUMPS library)
- Memory requirements increase rapidly with problem size
- Runtime typically scales like \( f^{\uparrow d} \) in \( d \) dimensions

High-order FEM

- **Hierarchical**
  - Hierarchical set of basis shape functions
  - Nodal, edge, face and bubble shape functions
  - Bubble shape functions can be eliminated from the global system (condensation)

- **Variable order**
  - The polynomial order $p$ can be adjusted in each element independently
  - A conformity rule is applied at inter-element boundaries (maximum or minimum rule)
2. Literature review and objectives
Previous studies on *p-FEM* Biot: Horlin et al. (2001, 2005)

- **Fixed order *p*-FEM**
  - \{U,u\} formulation
  - Fixed order $p_{\text{solid}} = p_{\text{fluid}}$
  - Hexahedral meshes only
  - Homogeneous foams and multi-layered media
- **Comparison of performance at fixed accuracy**
  - Successfully avoids locking and reduction of cost
  - $p=4$ is the optimal choice for reasonable accuracy


Previous studies on \( p\text{-FEM} \) Biot: Rigobert (2003)

- **Fixed order \( p\text{-FEM} \)**
  - \{U,\( p\}\} formulation
  - Different interpolation orders \( p_{\text{solid}} \neq p_{\text{fluid}} \)
  - Hexahedral meshes only
  - Homogeneous foams only

- **Conclusions**
  - The solid and fluid phase exhibit very different behaviors
  - Using different orders for each phase leads to dofs reduction
  - Analytical description is used to explain the behavior
  - Not fully decoupled: a sufficiently high order \( p_{\text{solid}} \) is required to reach convergence of the fluid phase indicator and vice-versa

Objective of this work

1) Re-examine the cost of $p$-FEM for Biot

2) Propose a simple error indicator to determine a-priori the order
3. Fixed-order cost analysis
Test Case description

- **Boundary conditions:**
  1) Surface pressure or traction BC
  2) Bonded to back wall
  3) Sliding condition on lateral faces

- **Material**
  - Polyurethane foam from Horlin 2005

- **Stack dimensions:** $4\text{cm}^3$
- **Fixed order:** $p_{\text{solid}} = p_{\text{fluid}}$
- **Frequency:** 10kHz
- **Condensation applied**
- **Mumps direct solver**
Memory required at 10kHz (polyurethane foam)

- Cost at fixed accuracy diminishes drastically for increasing values of $p$ at low orders ($p \leq 4$)
- For $p > 4$ the gain is less significant, but still visible
- Static condensation plays an important role
- Similar trends observed for other element types (HEXA, PRISM)
4. Variable order strategy
Automatic a-priori adaptivity workflow

- Before each **Frequency**
- Scan all 3D Biot elements
  1) Get the local **Material properties**
  2) Get the Biot wavenumbers $k_1, k_2$ and $(k_3)$
  3) For each edge:
     - Solve a single element 1D problem for $k_1, k_2$ $(k_3)$
     - Increase the order $p$ until below **Target accuracy**
  4) Apply conformity rules to define face and element orders across the full mesh
  5) Assemble and solve

Accuracy of $p$-FEM in the $kh$ complex plane

- **Biot formulations**: $kh$ is complex (for all types of waves)
- **Influence on accuracy**
  - Single 1D element $p$-FEM
  - $p=1$ to $p=9$
  - $L^2$ error is measured
  - $\Re(kh) \in [-10;10], \Im(kh) \in [-10;10]$
- **Conclusion**: Real and imaginary part of $kh$ need to be taken into account
5. First Results
Test case – complex structural excitation

- 10cm x 10cm x 2cm stack
- Polyurethane foam (Horlin 2005)
- 3 element types
- Frequency range [20, 2000] Hz

- **Bottom Face:** Traction BC
- **Top Face:** No radiation condition
- **Side faces:** sliding wall condition
Results at probe point

- Only the compression wavelengths \((k_{↓1}, k_{↓2})\)

\[ p_{↓\text{solid}} = \max (p_{↓\text{ind}}(k_{↓1}), p_{↓\text{ind}}(k_{↓2}), 2) \]

\[ p_{↓\text{fluid}} = \min (p_{↓\text{ind}}(k_{↓1})) \]
Results at probe point - $p_{\text{fluid}} = p_{\text{solid}}$

- Here we enforce $p_{\text{fluid}} = p_{\text{solid}}$
- Similar results
What is the gain in reducing \( p_{\text{fluid}} \)?

- Significant gains in using a different order for the fluid phase (~25-30% reduction in memory)
- HEXA and PRISM elements more efficient than the TETRA
6. Conclusion and future work
Conclusions

- Static condensation should be included
- No drastic cost reduction for $p > 4$ but still a positive impact of order increase
- Investigated a simple error indicator to determine the order in each element
- Error is not well controlled but overall, the dynamics is still captured
- Significant gain in using a lower order for the fluid phase

Future work

- New error indicator, with a single element solving 1D Biot equations instead of 1D Helmholtz with complex wavenumber
- 3D case with analytical solution (periodic BC)
- Test other materials and other configurations (multi-layered media)