Nonlocal Maxwellian Theory of Sound Propagation in Fluid-Saturated Rigid-Framed Porous Media*

* Lafarge & Nemati, Wave Motion (2013)

Checking the theory* in a simple 2D periodic geometry:

(local and Bragg resonances)

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Outline

- Introduction

Nonlocal Maxwellian theory
  - Macroscopic equations
  - Upscaling procedures

- Validation
Introduction: a macroscopic theory

A macroscopic theory describing sound propagation through porous media

- Unbounded saturated porous media: fluid-solid
  - Solid is rigid
  - Fluid is viscothermal

- **Local theory** (Classical Equivalent-Fluid)
  - Wavelength $\lambda \gg L$
  - Microscopic scale: the fluid is considered to be incompressible $\Rightarrow \nabla \cdot \mathbf{v} = 0$
  - It gives only the first normal mode
  - Local theory is not complete...

**Generalization:** Nonlocal theory

- Temporal dispersion + spatial dispersion
- $\nabla \cdot \mathbf{v} \neq 0$
- Nonasymptotic homogenization
- Beyond the long-wavelength limit
Viscothermal fluid equations for a small perturbation + interface conditions

- In the viscothermal fluid
  
  **Mass balance:** \( \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{v} = 0 \)

  **Momentum balance:** \( \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \eta \nabla^2 \mathbf{v} + (\zeta + \frac{1}{3} \eta) \nabla (\nabla \cdot \mathbf{v}) \)

  **Energy balance:** \( \rho_0 c_p \frac{\partial T}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \nabla^2 T \)

  **State:** \( \gamma \chi_0 \rho = b + \beta_0 \tau \)

- At the fluid-solid interface
  
  \( \sigma = 0, \quad \tau = 0 \)

  \( \epsilon \approx 0.01 \ll 1 \)
The « Lorentz » fields are the average of microscopic fields:

\[ E = \langle e \rangle \quad B = \langle b \rangle \]

\[ E \times H = \text{electromagnetic part of energy current density} \]
Theory: macroscopic acoustic equations

Maxwellian acoustic equations

**Field equations**

\[
\frac{\partial B}{\partial t} + \nabla \cdot \mathbf{V} = 0
\]

\[
\frac{\partial \mathbf{D}}{\partial t} = -\nabla H
\]

**Constitutive relations**

\[
\mathbf{D} = \hat{\rho} \mathbf{V}
\]

\[
H = \hat{\chi}^{-1} B
\]

- **Temporal dispersion (Local)**

\[
\mathbf{D}(t, r) = \int_{-\infty}^{t} dt' \rho(t - t') \mathbf{V}(t', r) \Rightarrow \mathbf{D}(\omega, r) = \rho(\omega) \mathbf{V}(\omega, r)
\]

\[
H(t, r) = \int_{-\infty}^{t} dt' \chi^{-1}(t - t') B(t', r) \Rightarrow H(\omega, r) = \chi^{-1}(\omega) B(\omega, r)
\]

- **Temporal dispersion + Spatial dispersion (Nonlocal)**

\[
\mathbf{D}(t, r) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \rho(t - t', \mathbf{r} - \mathbf{r}') \mathbf{V}(t', \mathbf{r}') \Rightarrow \mathbf{D}(\omega, \mathbf{k}) = \rho(\omega, \mathbf{k}) \mathbf{V}(\omega, \mathbf{k})
\]

\[
H(t, r) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \chi^{-1}(t - t', \mathbf{r} - \mathbf{r}') B(t', \mathbf{r}') \Rightarrow H(\omega, \mathbf{k}) = \chi^{-1}(\omega, \mathbf{k}) B(\omega, \mathbf{k})
\]

\[
\mathbf{VH} = \text{acoustic part of energy current density} = \langle p \mathbf{V} \rangle
\]

\[
\mathbf{V} = \langle \mathbf{v} \rangle \quad \mathbf{B} = \langle \mathbf{b} \rangle
\]
In the long-wavelength regime $\lambda \gg L$ and in the limit $\nabla \cdot \mathbf{v} = 0$:

- Spatial nonlocality is simply ignored, the time nonlocality is not completely described.
- Time nonlocality originates only in dissipative processes that occur with delays.
- If we remove the losses and assume local behavior ($\nabla \cdot \mathbf{v} = 0$ at the pore scale) $\implies$ Response of the fluid to an excitation should be instantaneous.
- $\rho(t - t') = \rho_0 \alpha_\infty \delta(t - t')$ and $\chi^{-1}(t - t') = \chi^{-1}_0 \delta(t - t')$
- There is no temporal dispersion ...

The dispersion is wholly linked to the losses!
Upscaling procedures

- Averaging in nonlocal theory is ensemble averaging
- **Periodic medium**: ensemble of random translations $\rightarrow$ Cell average
  - $\mathbf{V} = \langle \mathbf{v} \rangle$, $\mathbf{B} = \langle \mathbf{b} \rangle \equiv \langle \rho' / \rho_0 \rangle$

**Local**

- Macroscopic pressure $P = \langle p \rangle$
- Equivalent Fluid

**Nonlocal**

- Macroscopic pressure: $H \langle \mathbf{v} \rangle = \langle p \mathbf{v} \rangle$
- Generalized Equivalent Fluid

\[
\begin{array}{ll}
\rho(\omega), & \chi(\omega) \\
\text{One wave} & \\
\end{array}
\]

\[
\begin{array}{ll}
\rho(\omega, \mathbf{k}), & \chi(\omega, \mathbf{k}) \\
\text{Several waves} & \\
\end{array}
\]
FORCE-CURRENT ANALOGY

\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \]

\[ \frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} + \mathbf{J} \]

\[ \mathbf{D} = \epsilon \mathbf{E} \]

\[ \mathbf{H} = \mu^{-1} \mathbf{B} \]

\[ \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \mathbf{V} = 0 \]

\[ \frac{\partial \mathbf{D}}{\partial t} = -\nabla \mathbf{H} + \mathbf{F} = -\nabla \varphi \]

\[ \mathbf{D} = \rho \mathbf{V} \]

\[ \mathbf{H} = \chi^{-1} \mathbf{B} \]
### Upscaling procedures

- **In the visco-thermal fluid**
  
  \[
  \frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0
  \]

  \[
  \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \eta \nabla^2 \mathbf{v} + \left( \zeta + \frac{1}{3} \eta \right) \nabla (\nabla \cdot \mathbf{v}) - \nabla (P e^{ikx-i\omega t})
  \]

  Added for determination of density

  \[
  \rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \kappa \nabla^2 \tau
  \]

 Added for determination of bulk modulus

  \[
  \gamma \chi_0 p = b + \beta_0 \tau
  \]

- **On the fluid-solid interface:**

  \[\mathbf{v} = 0, \quad \tau = 0\]

  Effective density

  \[
  \rho(\omega, k) = \frac{k (P + P(\omega, k))}{\omega \langle v(\omega, k, r) \rangle}
  \]

  where \( P \langle \mathbf{v} \rangle = \langle p \mathbf{v} \rangle \)

  Effective modulus

  \[
  \chi^{-1}(\omega, k) = \frac{P(\omega, k) + P}{\langle b(\omega, k, r) \rangle}
  \]

Lafarge & Nemati, Wave Motion (2013)

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Direct consequences of the FORCE-CURRENT analogy!
Wavenumbers, effective densities and compressibilities

- For each frequency, the complex wavenumbers, constants of the medium, are the solution of the nonlocal dispersion equation:

\[ \rho(\omega, k) \chi(\omega, k) \omega^2 = k^2 \implies k_n(\omega), \ n = 1, 2, \ldots \]

\[ \rho(\omega, k_n(\omega)) = \rho_n(\omega), \text{ and } \chi(\omega, k_n(\omega)) = \chi_n(\omega) \]

Homogenized medium

- Free homogeneous fluid

\[ \rho_0 \chi_0 c_0^2 = 1 \]

- Local Effective Fluid

\[ \rho(\omega) \chi(\omega) c^2(\omega) = 1 \]

- Nonlocal Effective Fluid

\[ \rho_1(\omega) \chi_1(\omega) c_1^2(\omega) = 1 \]

\[ \rho_2(\omega) \chi_2(\omega) c_2^2(\omega) = 1 \]

\[ \ldots \]

\[ \rho_n(\omega) \chi_n(\omega) c_n^2(\omega) = 1 \]

\[ \ldots \]
A SIMPLE GEOMETRY FOR QUASI ANALYTICAL VERIFICATION:

Bloch modes:

$L = 2.0 \text{cm}, \quad d = 1.0 \text{cm}, \quad h_1 = 1.0 \text{cm}, \quad h_2 = 9.86 \text{cm}$

Nonlocal theory:
Solving through Newton scheme the nonlocal dispersion equation

Direct Bloch calculation, 1st Brillouin Zone

$\frac{h_2}{h_1} = 10$
Higher orders modes: they remain in the first Brillouin zone. No difference between Bloch and Nonlocal calculations.

Generalisation of the notion of tortuosity. .... mode 1.
Generalisation of the notion of compressibility .... mode 1
Generalisation of the notion of tortuosity

Generalisation of the notion of compressibility

mode 2

mode 2
Conclusions

- Establishment of a general theory allowing for spatial dispersion and full and untruncated temporal dispersion
- The role of the velocity divergence (or constant pressure) is clarified
- Metamaterial (metafluid) behavior is a natural consequence of a more general theory taking into account the full nonlocal effects
- Effective properties of phononic crystals can be described in Bragg regime
- Generalization to bounded medium and arbitrary direction of propagation
- Generalization to elastic solid and composite medium