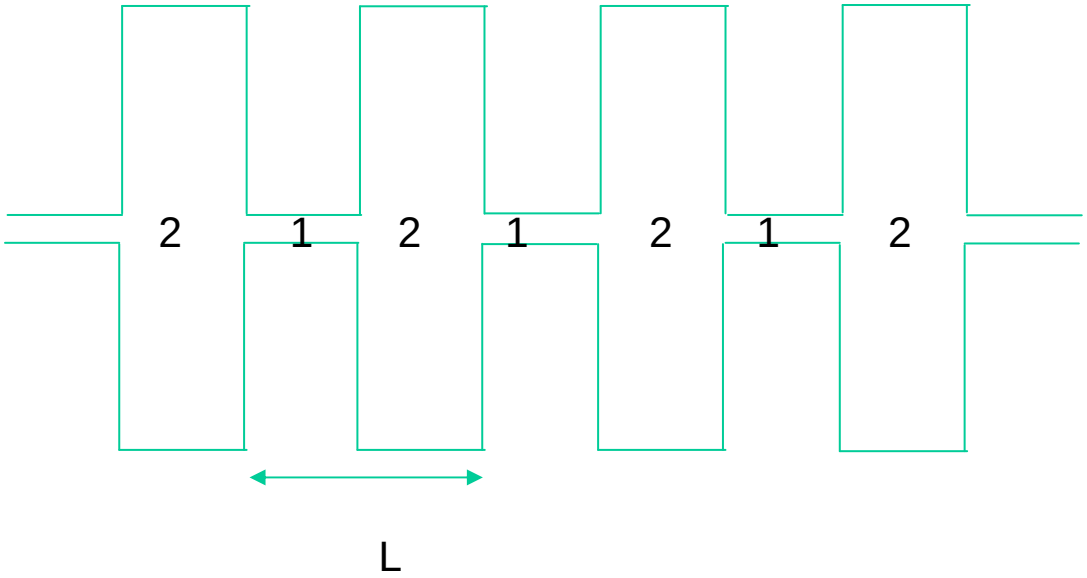
*Nonlocal Maxwellian Theory of Sound Propagation in Fluid-Saturated Rigid-Framed Porous Media**

*Lafarge & Nemati, Wave Motion (2013)

Checking the theory* in a simple 2D periodic geometry :



(local and Bragg resonances)

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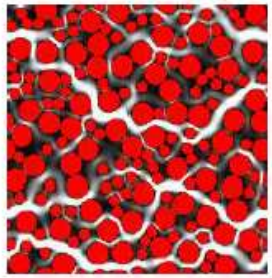
Outline

- Introduction
- Nonlocal Maxwellian theory
  - Macroscopic equations
  - Upscaling procedures
- Validation

Introduction: a macroscopic theory

A macroscopic theory describing sound propagation through porous media

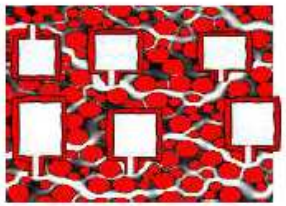
- Unbounded saturated porous media: fluid-solid
  - Solid is rigid
  - Fluid is viscothermal
- Local theory **(Classical Equivalent-Fluid)**
  - Wavelength $\lambda \gg L$
  - Microscopic scale: the fluid is considered to be incompressible $\Rightarrow \nabla.\mathbf{v} = 0$
  - It gives only the first normal mode
  - Local theory is not complete...



Volume average

Generalization: Nonlocal theory

- Temporal dispersion + spatial dispersion
- $\nabla.\mathbf{v} \neq 0$
- Nonasymptotic homogenization
- Beyond the long-wavelength limit



Ensemble average

Viscothermal fluid equations for a small perturbation + interface conditions

- In the viscothermal fluid

Mass balance: $$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0$$

Momentum balance: $$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \cancel{\eta \nabla^2 \mathbf{v} + \left(\zeta + \tfrac{1}{3}\eta\right) \nabla (\nabla \cdot \mathbf{v})}$$

Energy balance: $$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \cancel{\kappa \nabla^2 \tau}$$

State: $$\gamma \chi_0 p = b + \beta_0 \tau$$

$\rho_0 (1 + i\varepsilon)$

$\varepsilon \approx 0.01 << 1$

- At the fluid-solid interface

$$\mathbf{v} = 0, \quad \tau = 0$$

Theory: Macroscopic electromagnetic equations

Maxwell equations

Field equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$
$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H}$$

Constitutive relations

$$\mathbf{D} = \hat{\epsilon}\,\mathbf{E}$$
$$\mathbf{H} = \hat{\mu}^{-1}\mathbf{B}$$

Maxwell

Lorentz

- Temporal dispersion (Local)

$$\mathbf{D}(t,\mathbf{r}) = \int_{-\infty}^{t} dt'\ \epsilon(t-t')\mathbf{E}(t',\mathbf{r}) \quad \Rightarrow \mathbf{D}(\omega,\mathbf{r}) = \epsilon(\omega)\mathbf{E}(\omega,\mathbf{r})$$
$$\mathbf{H}(t,\mathbf{r}) = \int_{-\infty}^{t} dt'\ \mu^{-1}(t-t')\mathbf{B}(t',\mathbf{r}) \Rightarrow \mathbf{H}(\omega,\mathbf{r}) = \mu^{-1}(\omega)\mathbf{B}(\omega,\mathbf{r})$$

- Temporal dispersion + Spatial dispersion (Nonlocal)

$$\mathbf{D}(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \int d\mathbf{r}'\epsilon(t-t',\mathbf{r}-\mathbf{r}')\mathbf{E}(t',\mathbf{r}') \quad \Rightarrow \mathbf{D}(\omega,\mathbf{k}) = \epsilon(\omega,\mathbf{k})\mathbf{E}(\omega,\mathbf{k})$$
$$\mathbf{H}(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \int d\mathbf{r}'\mu^{-1}(t-t',\mathbf{r}-\mathbf{r}')\mathbf{B}(t',\mathbf{r}') \Rightarrow \mathbf{H}(\omega,\mathbf{k}) = \mu^{-1}(\omega,\mathbf{k})\mathbf{B}(\omega,\mathbf{k})$$

The « Lorentz » fields are the average of microscopic fields :

$$\mathbf{E} = \langle \mathbf{e} \rangle \qquad \mathbf{B} = \langle \mathbf{b} \rangle$$

« Maxwell » = ?

$\mathbf{E}\times\mathbf{H} = \textit{electromagnetic part of energy current density}$

Theory: macroscopic acoustic equations

Maxwellian acoustic equations

Field equations

$$\frac{\partial B}{\partial t} + \nabla \cdot \mathbf{V} = 0$$
$$\frac{\partial \mathbf{D}}{\partial t} = -\nabla H$$

Constitutive relations

$$\mathbf{D} = \hat{\rho}\,\mathbf{V}$$
$$H = \hat{\chi}^{-1} B$$

Maxwell

Lorentz

- Temporal dispersion (Local)

$$\mathbf{D}(t,\mathbf{r}) = \int_{-\infty}^{t} dt'\ \rho(t-t')\mathbf{V}(t',\mathbf{r}) \quad \Rightarrow \mathbf{D}(\omega,\mathbf{r}) = \rho(\omega)\mathbf{V}(\omega,\mathbf{r})$$
$$H(t,\mathbf{r}) = \int_{-\infty}^{t} dt'\ \chi^{-1}(t-t')B(t',\mathbf{r}) \Rightarrow H(\omega,\mathbf{r}) = \chi^{-1}(\omega)B(\omega,\mathbf{r})$$

- Temporal dispersion + Spatial dispersion (Nonlocal)

$$\mathbf{D}(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \int d\mathbf{r}'\rho(t-t',\mathbf{r}-\mathbf{r}')\mathbf{V}(t',\mathbf{r}') \quad \Rightarrow \mathbf{D}(\omega,\mathbf{k}) = \rho(\omega,\mathbf{k})\mathbf{V}(\omega,\mathbf{k})$$
$$H(t,\mathbf{r}) = \int_{-\infty}^{t} dt' \int d\mathbf{r}'\chi^{-1}(t-t',\mathbf{r}-\mathbf{r}')B(t',\mathbf{r}') \Rightarrow H(\omega,\mathbf{k}) = \chi^{-1}(\omega,\mathbf{k})B(\omega,\mathbf{k})$$

$\mathbf{V}H = \textit{acoustic part of energy current density} = \langle p\mathbf{v}\rangle$ $\qquad \mathbf{V} = \langle \mathbf{v}\rangle \qquad B = \langle b\rangle$

Local theory and temporal dispersion

In the long-wavelength regime $\lambda \gg L$ and in the limit $\nabla.\mathbf{v} = 0$:

- Spatial nonlocality is simply ignored, the time nonlocality is not completely described.
- Time nonlocality originates only in dissipative processes that occur with delays.
- If we remove the losses and assume local behavior ($\nabla.\mathbf{v} = 0$ at the pore scale) $\Longrightarrow$ Response of the fluid to an excitation should be instantaneous.
- $\rho(t - t') = \rho_0 \alpha_\infty \delta(t - t')$ and $\chi^{-1}(t - t') = \chi_0^{-1} \delta(t - t')$
- There is no temporal dispersion ...

The dispersion is wholly linked to the losses!

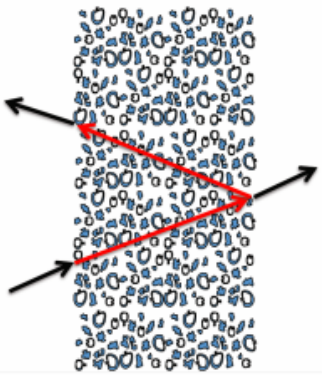
Upscaling procedures

- Averaging in nonlocal theory is ensemble averaging
- Periodic medium: ensemble of random translations $\Longrightarrow$ Cell average
- $\mathbf{V} = \langle \mathbf{v} \rangle$, $B = \langle b \rangle \equiv \langle \rho' / \rho_0 \rangle$

<u>Local</u>

- Macroscopic pressure $P = \langle p \rangle$
- Equivalent Fluid

$$\boxed{\rho(\omega), \quad \chi(\omega)}$$

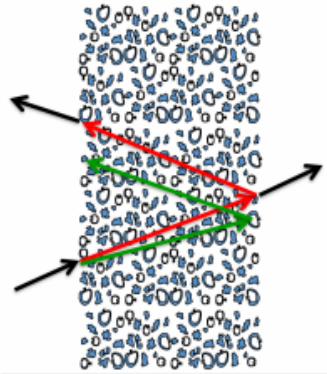


One wave

<u>Nonlocal</u>

- Macroscopic pressure: $H\langle \mathbf{v} \rangle = \langle p\mathbf{v} \rangle$
- Generalized Equivalent Fluid

$$\boxed{\rho(\omega, \mathbf{k}), \quad \chi(\omega, \mathbf{k})}$$



Several waves

FORCE-CURRENT ANALOGY

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$
$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} \quad + \mathbf{J}$$

$$\mathbf{D} = \hat{\epsilon}\, \mathbf{E}$$

$$\mathbf{H} = \hat{\mu}^{-1} \mathbf{B}$$

$$\frac{\partial B}{\partial t} + \nabla \cdot \mathbf{V} = 0$$
$$\frac{\partial \mathbf{D}}{\partial t} = -\nabla H + \mathbf{F}(= -\partial \wp)$$

$$\mathbf{D} = \hat{\rho}\, \mathbf{V}$$

$$H = \hat{\chi}^{-1} B$$

Upscaling procedures

- In the visco-thermal fluid

$$\frac{\partial b}{\partial t} + \nabla \cdot \mathbf{v} = 0$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \cancel{\eta \nabla^2 \mathbf{v} + \left(\zeta + \frac{1}{3}\eta\right) \nabla (\nabla \cdot \mathbf{v})} - \underbrace{\nabla \left(\mathcal{P} e^{ikx - i\omega t}\right)}_{\text{Added for determination of density}}$$

$$\rho_0 c_p \frac{\partial \tau}{\partial t} = \beta_0 T_0 \frac{\partial p}{\partial t} + \cancel{\kappa \nabla^2 \tau} + \underbrace{\frac{\partial}{\partial t} \left(\mathcal{P} e^{ikx - i\omega t}\right)}_{\text{Added for determination of bulk modulus}}$$

$$\gamma \chi_0 p = b + \beta_0 \tau$$

- On the fluid-solid interface: $\mathbf{v} = 0, \quad \tau = 0$

Effective density

$$\rho(\omega, k) = \frac{k\left(\mathcal{P} + P(\omega, k)\right)}{\omega \left\langle v(\omega, k, \mathbf{r}) \right\rangle}$$

where $P \left\langle \mathbf{v} \right\rangle = \left\langle p \mathbf{v} \right\rangle$

Effective modulus

$$\chi^{-1}(\omega, k) = \frac{P(\omega, k) + \mathcal{P}}{\left\langle b(\omega, k, \mathbf{r}) \right\rangle + \phi \gamma \chi_0 \mathcal{P}}$$

Lafarge & Nemati, Wave Motion (2013)

Direct consequences of the FORCE-CURRENT analogy !

Wavenumbers, effective densities and compressibilities

- For each frequency, the complex wavenumbers, constants of the medium, are the solution of the nonlocal dispersion equation:

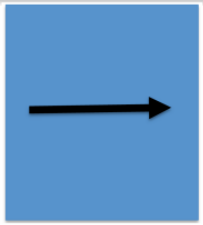
$$\boxed{\rho(\omega, k)\ \chi(\omega, k)\ \omega^2 = k^2} \implies k_n(\omega),\ n = 1, 2, \ldots$$

$$\boxed{\rho\ (\omega, k_n(\omega)) = \rho_n(\omega)}, \text{ and } \boxed{\chi\ (\omega, k_n(\omega)) = \chi_n(\omega)}$$

Homogenized medium

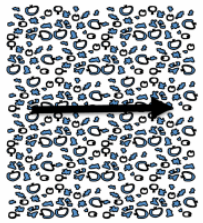
- Free homogeneous fluid

$$\boxed{\rho_0 \chi_0 c_0^2 = 1}$$



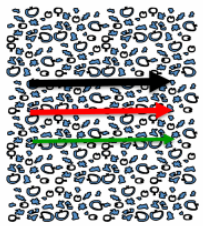
- Local Effective Fluid

$$\boxed{\rho(\omega)\ \chi(\omega)\ c^2(\omega) = 1}$$

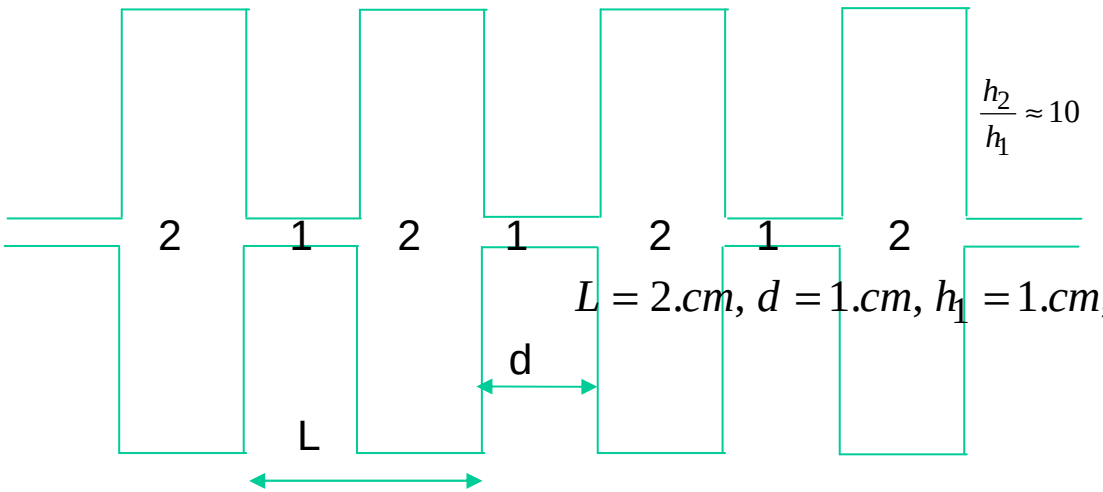


- Nonlocal Effective Fluid

$$\boxed{\begin{array}{c} \rho_1(\omega)\ \chi_1(\omega)\ c_1^2(\omega) = 1 \\ \rho_2(\omega)\ \chi_2(\omega)\ c_2^2(\omega) = 1 \\ \ldots \\ \rho_n(\omega)\ \chi_n(\omega)\ c_n^2(\omega) = 1 \\ \ldots \end{array}}$$



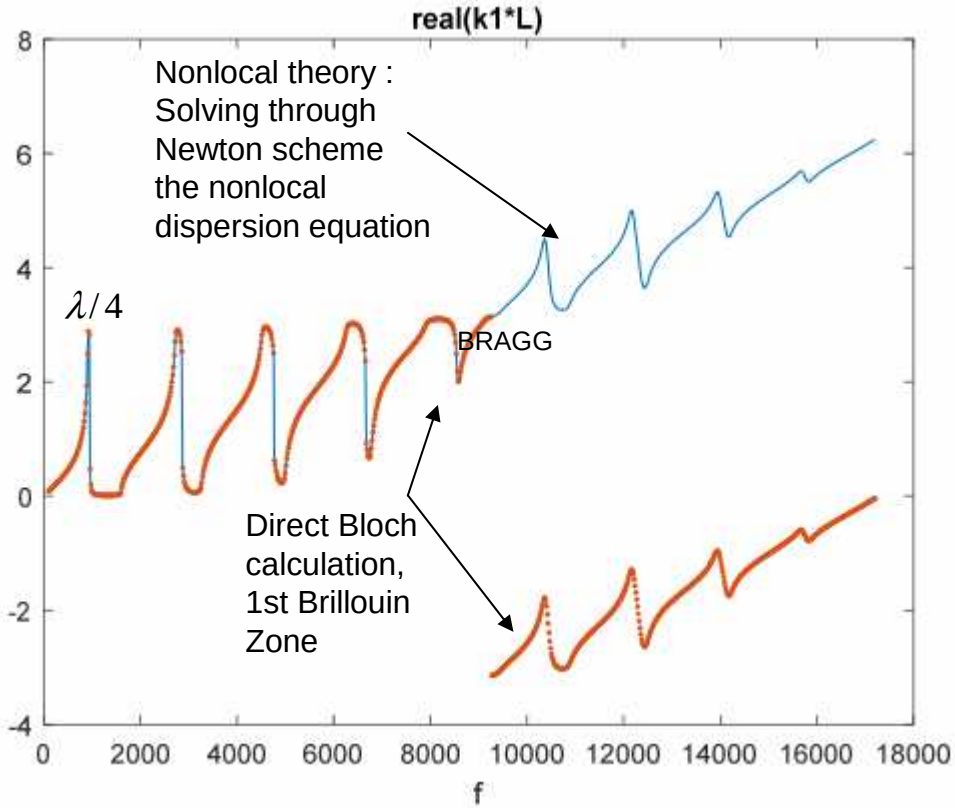
A SIMPLE GEOMETRY FOR QUASI ANALYTICAL VERIFICATION :

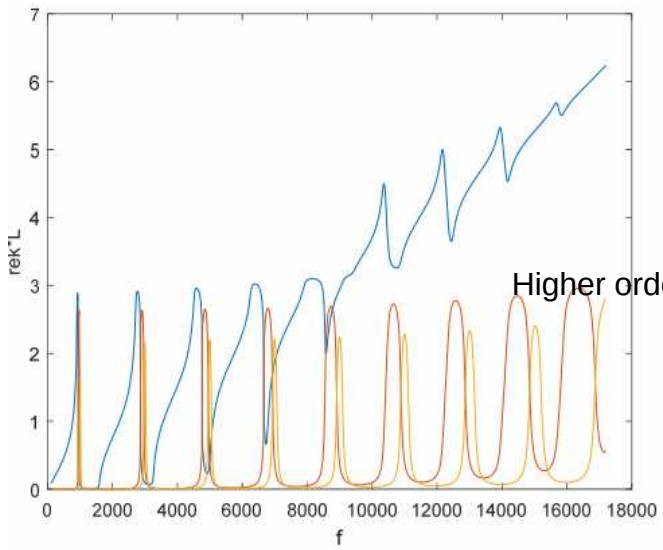


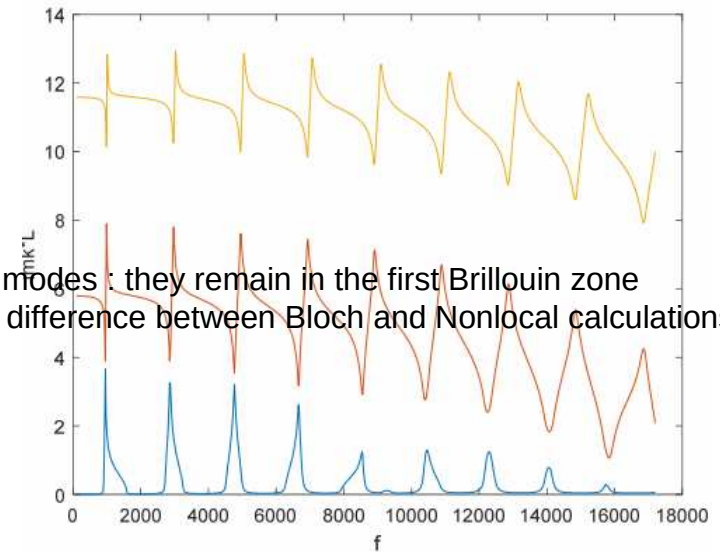
$$\frac{h_2}{h_1} \approx 10$$

$$L = 2.cm,\ d = 1.cm,\ h_1 = 1.cm,\ h_2 = 9.86cm$$

Bloch modes :



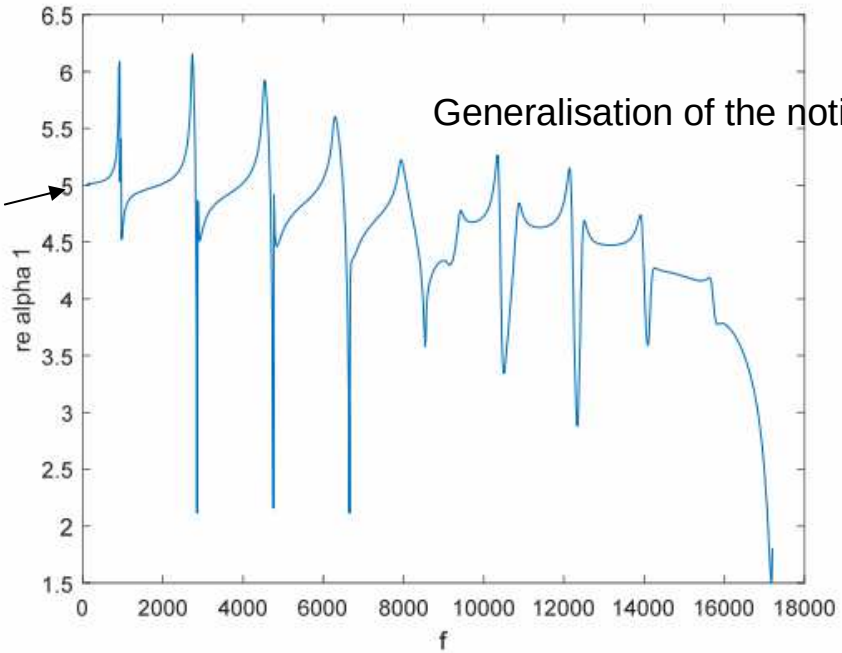


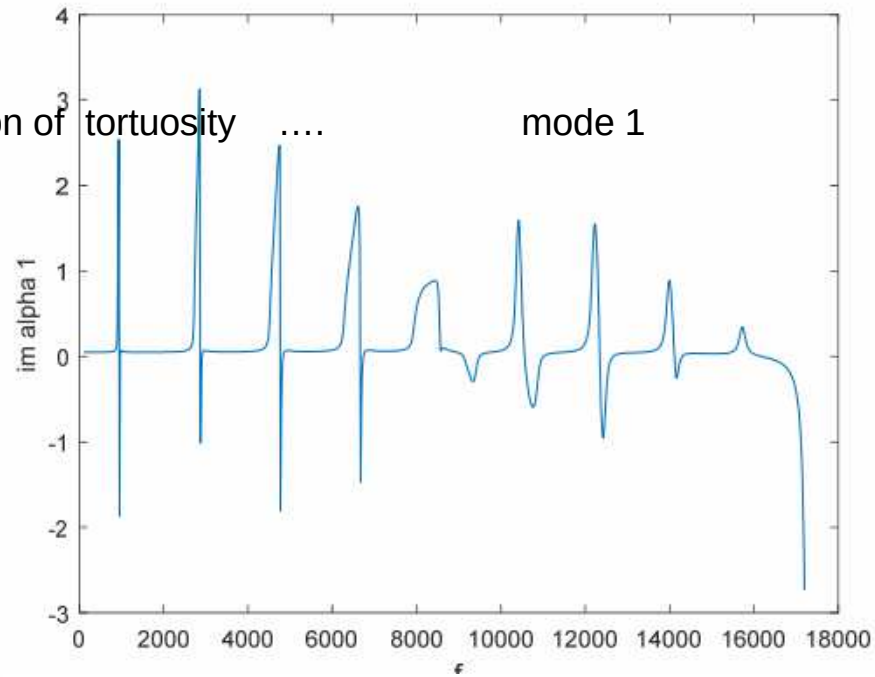


Higher orders modes : they remain in the first Brillouin zone
no difference between Bloch and Nonlocal calculations

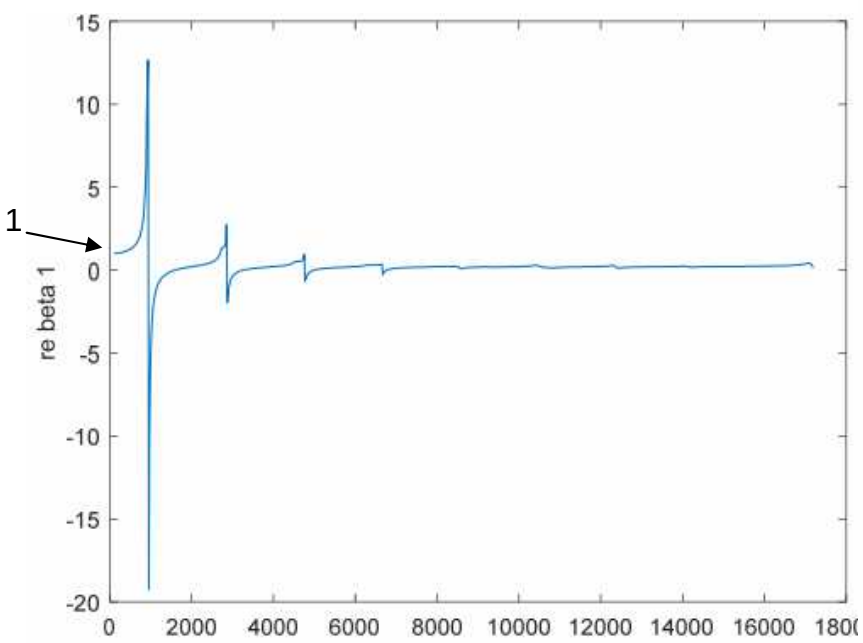
Generalisation of the notion of tortuosity …. mode 1

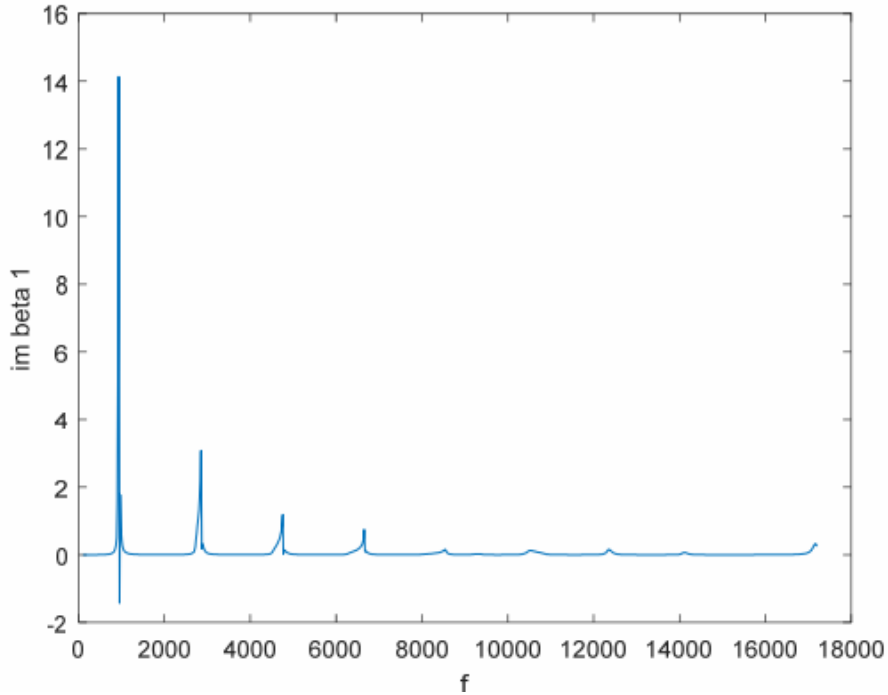
α_∞



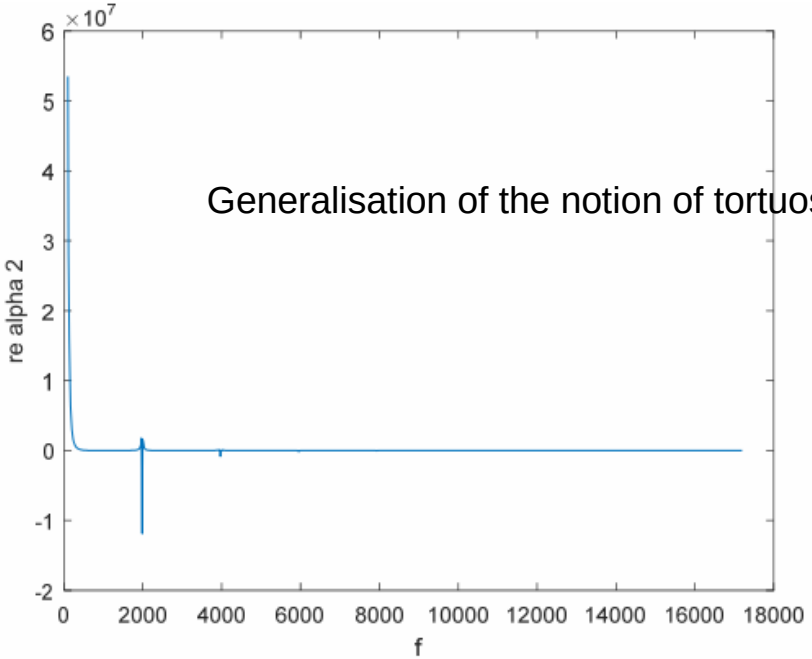


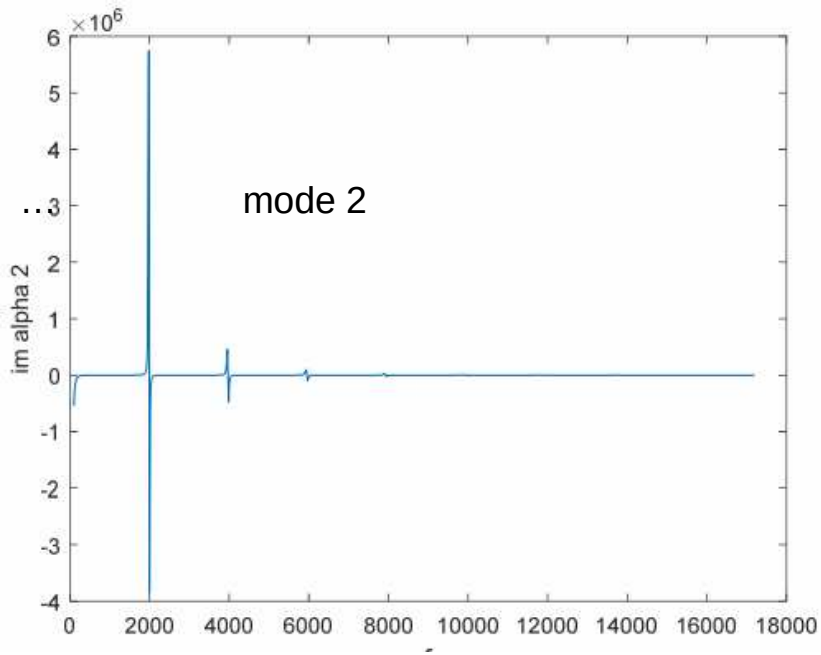
Generalisation of the notion of compressibility …. mode 1



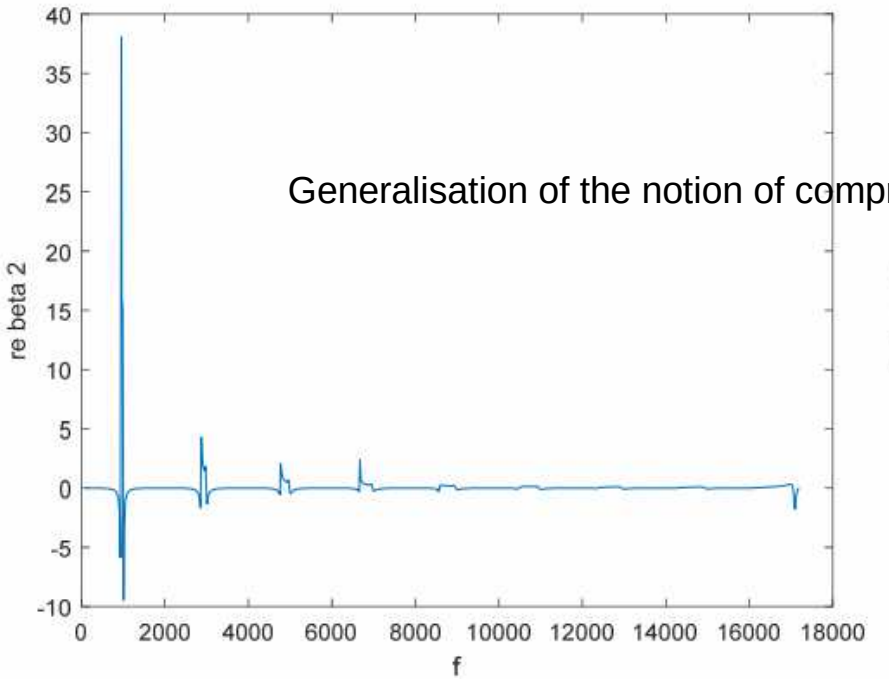


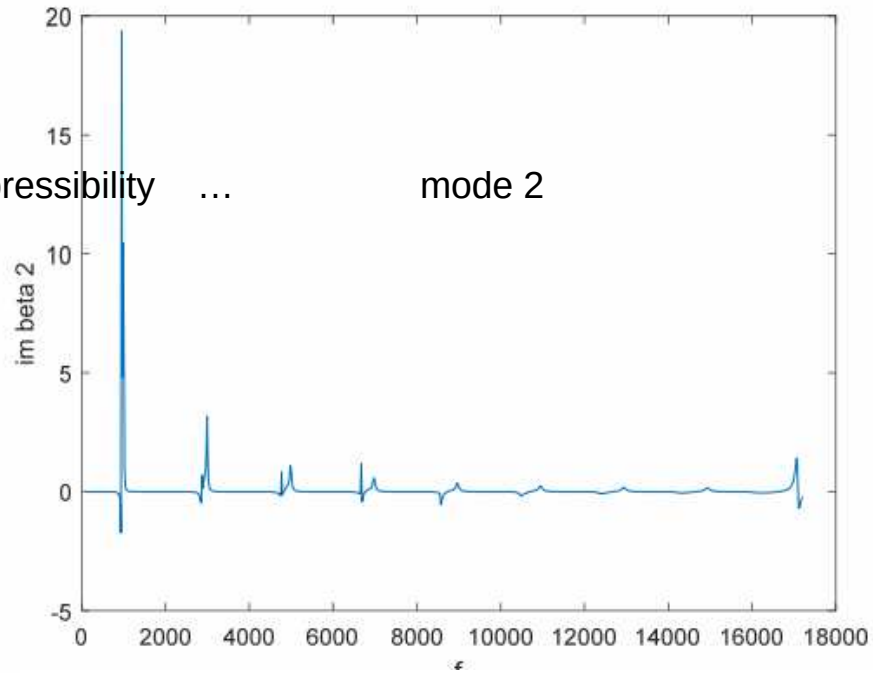
Generalisation of the notion of tortuosity … mode 2





Generalisation of the notion of compressibility … mode 2





Conclusions

- Establishment of a general theory allowing for spatial dispersion and full and untruncated temporal dispersion
- The role of the velocity divergence (or constant pressure) is clarified
- Metamaterial (metafluid) behavior is a natural consequence of a more general theory taking into account the full nonlocal effects
- Effective properties of phononic crystals can be described in Bragg regime
- Generalization to bounded medium and arbitrary direction of propagation
- Generalization to elastic solid and composite medium