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Adaptive high-order FEM for the mixed {U,p} poro-elastic equations

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Motivation

- Non expert users
- Commercial software – large applications
- Adjust the order automatically based on
  - Mesh characteristics
  - Frequency
  - Type of materials

Outline of the presentation

1. Biot formulation and p-FEM numerical model
2. Literature review and objectives
3. Fixed-order cost analysis
4. Adaptive order strategy
5. First results
6. Conclusions

1. Biot formulation and p-FEM numerical model

Weak form: Mixed {u,p} formulation

$$\int_\Omega \boldsymbol{\sigma}^s(\boldsymbol{u}) : \boldsymbol{\epsilon}^s(\boldsymbol{u}^*)\, d\Omega - \omega^2 \int_\Omega \tilde{\rho}\, \boldsymbol{u} \cdot \boldsymbol{u}^*\, \mathrm{d}\Omega$$
$$+ \int_\Omega \left[\frac{\phi^2}{\omega^2 \tilde{\rho}_{22}} \nabla p^* \cdot \nabla p - \frac{\phi^2}{R} p^* p \right] \mathrm{d}\Omega$$
$$- \int_\Omega (\gamma + \phi') \left(\nabla p^* \cdot \boldsymbol{u} + \boldsymbol{u}^* \cdot \nabla p \right) \mathrm{d}\Omega - \int_\Omega \phi' \left(p^* \nabla \cdot \boldsymbol{u} + p \nabla \cdot \boldsymbol{u}^* \right) \mathrm{d}\Omega$$
$$- \int_\Gamma \left(\boldsymbol{\sigma}^t \cdot \boldsymbol{n} \right) \cdot \boldsymbol{u}^*\, \mathrm{d}\Gamma - \int_\Gamma p^* \phi \left(\boldsymbol{U} - \boldsymbol{u} \right) \cdot \boldsymbol{n}\, \mathrm{d}\Gamma = 0.$$

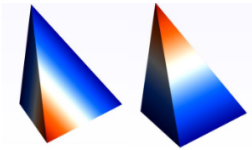
N. Atalla, M. A. Hamdi, R. Panneton, Enhanced weak integral formulation for the mixed poroelastic equations, Journal of Acoustics Society of America, 109, 2001.

- 4 unknowns in 3D (skeleton displacement and pressure)
- Direct Solver (MUMPS library)
- Memory requirements increase rapidly with problem size
- Runtime typically scales like $f^{\uparrow d}$ in d dimensions

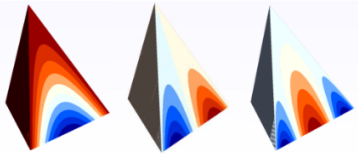
High-order FEM

- **Hierarchical**
  - Hierarchical set of basis shape functions
  - Nodal, edge, face and bubble shape functions
  - Bubble shape functions can be eliminated from the global system (condensation)

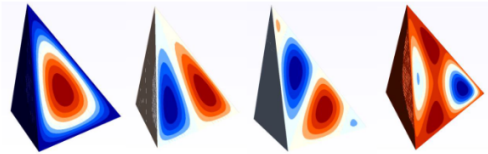
- **Variable order**
  - The polynomial order p can be adjusted in each element independently
  - A conformity rule is applied at inter-element boundaries (maximum or minimum rule)



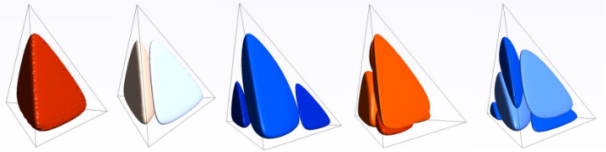
Vertex Shape Functions



Edge Shape Functions



Face Shape Functions



Bubble Shape Functions

2. Literature review and objectives

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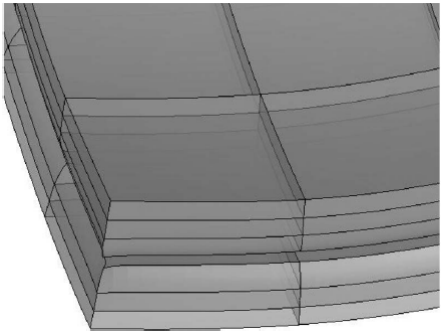
Previous studies on *p-FEM* Biot: Horlin et al. (2001, 2005)



- **Fixed order *p*-FEM**
  - {U,u} formulation
  - Fixed order $p\downarrow solid = p\downarrow fluid$
  - Hexahedral meshes only
  - Homogeneous foams and multi-layered media
- **Comparison of performance at fixed accuracy**
  - Successfully avoids locking and reduction of cost
  - $p=4$ is the optimal choice for reasonable accuracy

N.-E Hörlin, M. Nordström, P. Göransson A 3-D hierarchical FE formulation of Biot's equations for elasto-acoustic modelling of porous media, Journal of Sound and Vibration, 245 (4), 633-652, 2001.

N.-E Hörlin, 3D hierarchical hp-FEM applied to elasto-acoustic modelling of layered porous media, Journal of Sound and Vibration, 285 (1-2), 341-363, 2005.

Previous studies on *p-FEM* Biot: Rigobert (2003)

- **Fixed order *p*-FEM**
  - {U,p} formulation
  - Different interpolation orders $p\downarrow solid \neq p\downarrow fluid$
  - Hexahedral meshes only
  - Homogeneous foams only
- **Conclusions**
  - The solid and fluid phase exhibit very different behaviors
  - Using different orders for each phase leads to dofs reduction
  - Analytical description is used to explain the behavior
  - Not fully decoupled: a sufficiently high order $p\downarrow solid$ is required to reach convergence of the fluid phase indicator and vice-versa

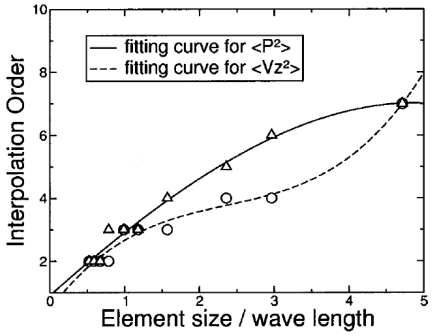


FIG. 6. Convergence criterion for the solid phase. Configuration (i). Given an element size for the mesh, an interpolation order in the solid phase that suffices for the convergence of $\langle P^2 \rangle$ (respectively, $\langle V_z^2 \rangle$) is determined by numerical experiments. △ (respectively, ○) stands for the corresponding couple of values interpolation order-element size to wavelength ratio.

S. Rigobert, N. Atalla, F. C. Sgard. Investigation of the convergence of the mixed displacement-pressure formulation for three-dimensional poroelastic materials using hierarchical elements, Journal of Acoustical Society of America, 114, 2607-2617, 2003.

Objective of this work

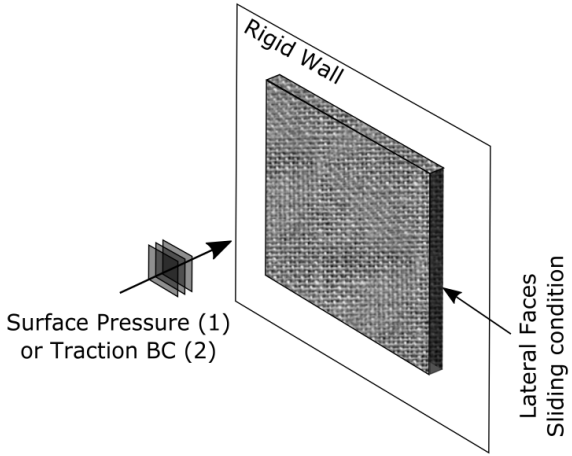
1) Re-examine the cost of *p-FEM* for Biot

2) Propose a simple error indicator to determine a-priori the order

3. Fixed-order cost analysis

Test Case description

- **Boundary conditions:**
  1) Surface pressure or traction BC
  2) Bonded to back wall
  3) Sliding condition on lateral faces



- **Material**
  - Polyurethane foam from Horlin 2005

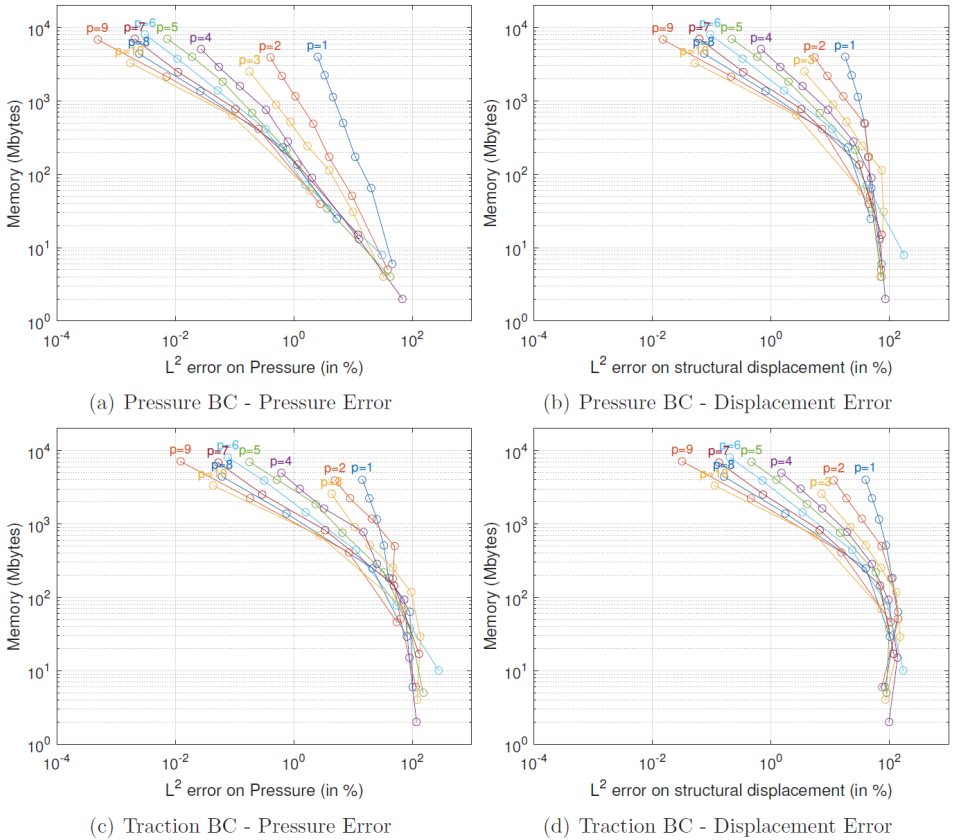
Material properties for a polyurethane foam

Static Young's modulus	70.0×10^3 Pa
Static Poisson's ratio	0.39
Porous material density	22.1 kg/m^3
Porosity	0.98
Fluid density	1.204 kg/m^3
Fluid dynamic viscosity	1.84×10^{-5}
Fluid ratio of specific heats	1.4
Prandtl number	0.71
Viscous characteristic length	1.1×10^{-4} m
Thermal characteristic length	7.42×10^{-4} m
Static flow resistivity	3.75×10^3 kg/m^3 s
Tortuosity	1.17
Thermal form factor	0.25
Gas constant (for air)	286.7 m^2/s^2 K
Absolute temperature	293.15 K

- **Stack dimensions:** $4cm↑3$
- **Fixed order:** $p↓solid = p↓fluid$
- **Frequency:** 10kHz
- Condensation applied
- Mumps direct solver

Memory required at 10kHz (polyurethane foam)

- Cost at fixed accuracy diminishes drastically for increasing values of p at low orders ($p \leq 4$)
- For $p > 4$ the gain is less significant, but still visible
- Static condensation plays an important role
- Similar trends observed for other element types (HEXA, PRISM)



(a) Pressure BC - Pressure Error

(b) Pressure BC - Displacement Error

(c) Traction BC - Pressure Error

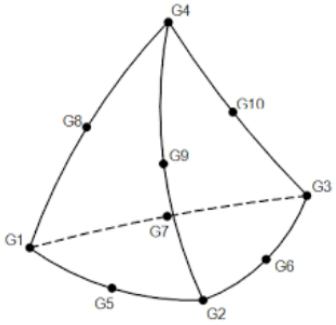
(d) Traction BC - Displacement Error

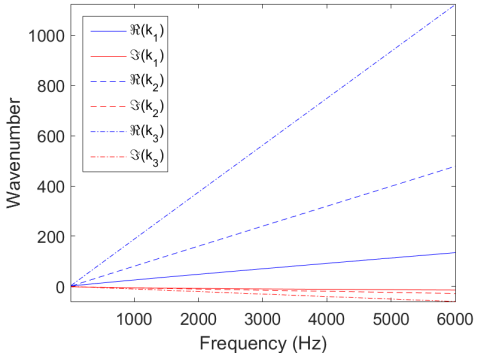
4. Variable order strategy

Automatic a-priori adaptivity workflow

- Before each **Frequency**
- Scan all 3D Biot elements

1) Get the local **Material properties**
2) Get the Biot wavenumbers k_1, k_2 and (k_3)
3) For each edge:
   - Solve a single element 1D problem for k_1, k_2 (k_3)
   - Increase the order p until below **Target accuracy**
4) Apply conformity rules to define face and element orders across the full mesh
5) Assemble and solve





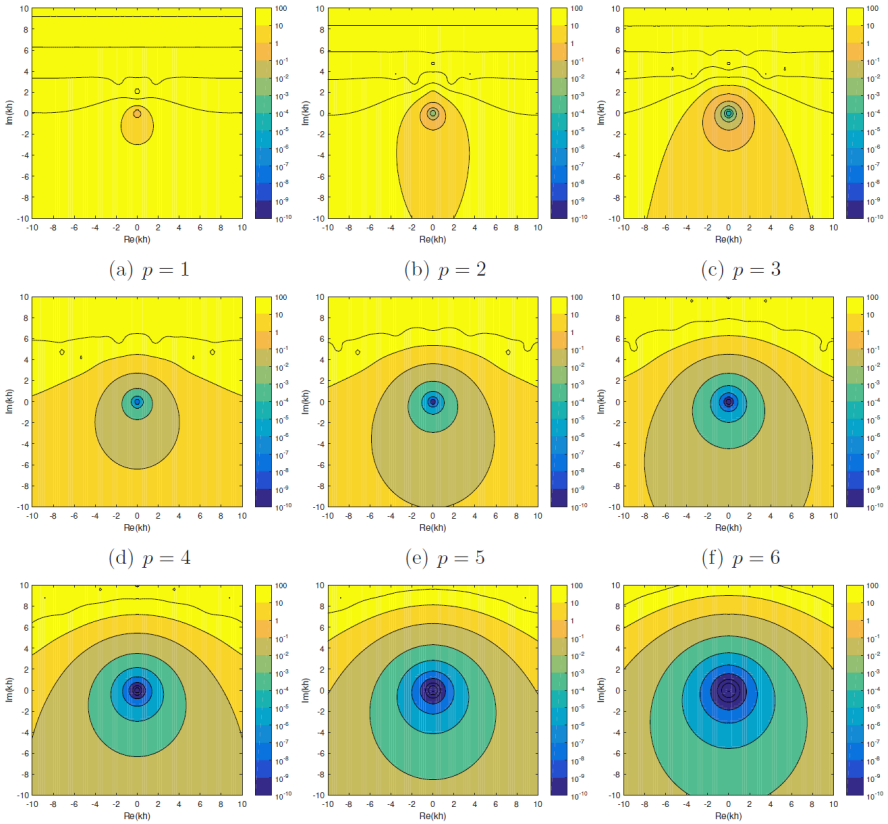
Bériot, H., Prinn, A., & Gabard, G. (2016). Efficient implementation of high-order finite elements for Helmholtz problems. *International Journal for Numerical Methods in Engineering*, *106*(3), 213-240.

Accuracy of *p*-FEM in the *kh* complex plane

- **Biot formulations:** kh is complex (for all types of waves)

- **Influence on accuracy**
  - Single 1D element p-FEM
  - p=1 to p=9
  - $L\hat{1}2$ error is measured
  - $\Re(kh)\in[-10;10], \Im(kh)\in[-10;10]$

- **<u>Conclusion:</u>** Real and imaginary part of kh need to be taken into account



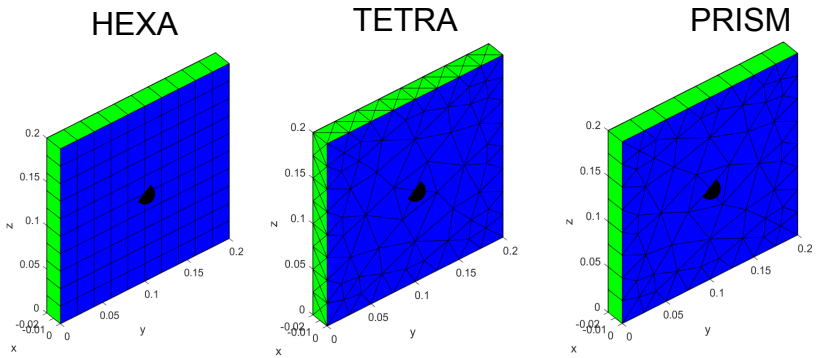
(a) $p = 1$ (b) $p = 2$ (c) $p = 3$

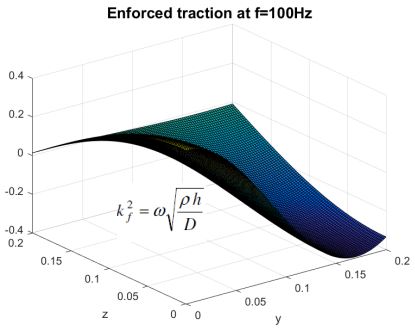
(d) $p = 4$ (e) $p = 5$ (f) $p = 6$

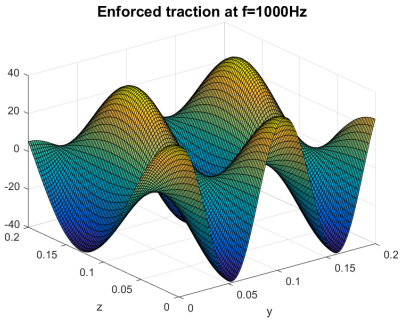
(g) $p = 7$ (h) $p = 8$ (i) $p = 9$

5. First Results

Test case – complex structural excitation







- 10cm x10cm x 2cm stack
- Polyurethane foam (Horlin 2005)
- 3 element types
- Frequency range $[20, 2000]$ Hz

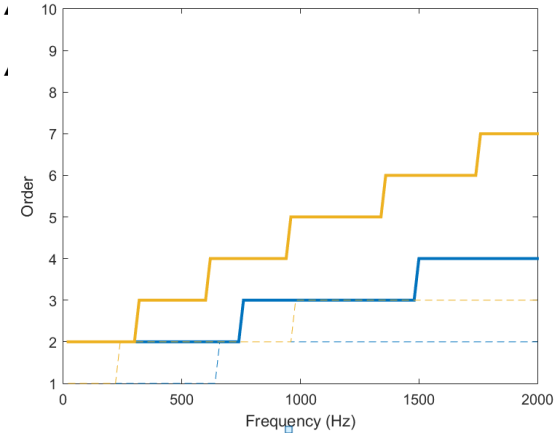
- **Bottom Face**: Traction BC
- **Top Face**: No radiation condition
- **Side faces**: sliding wall condition

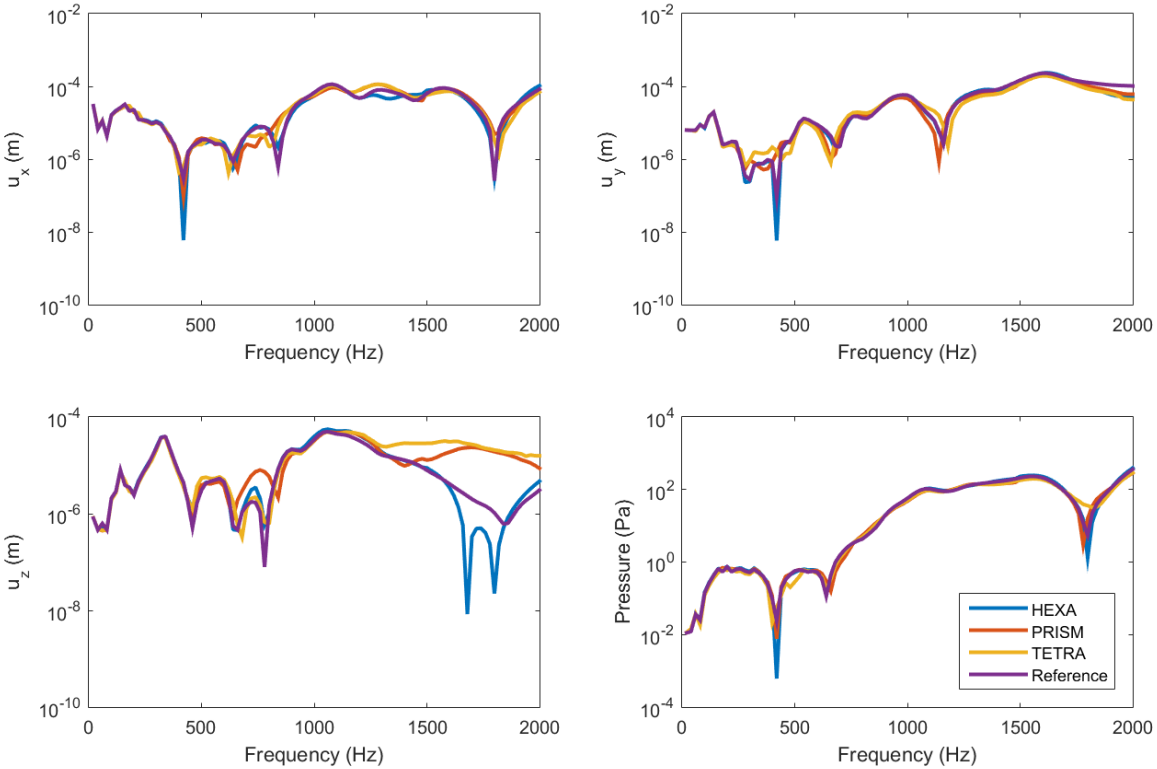
Results at probe point

- Only the compression wavelengths (k_1, k_2)

$$p_{solid} = max(p_{ind}(k_1), p_{ind}(k_2), 2)$$

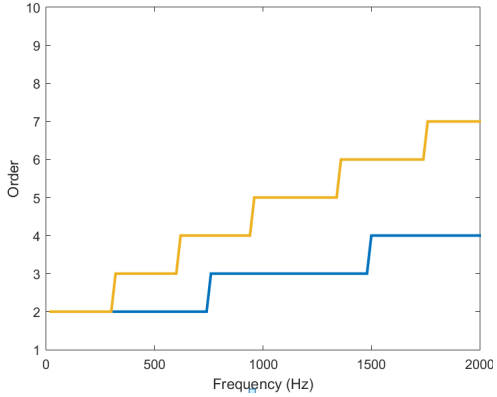
$$p_{fluid} = min(p_{ind}(k_1),$$

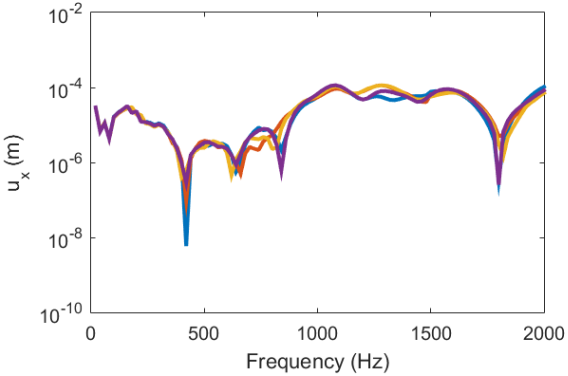


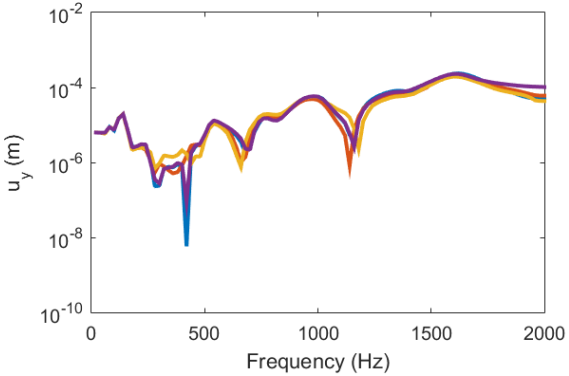


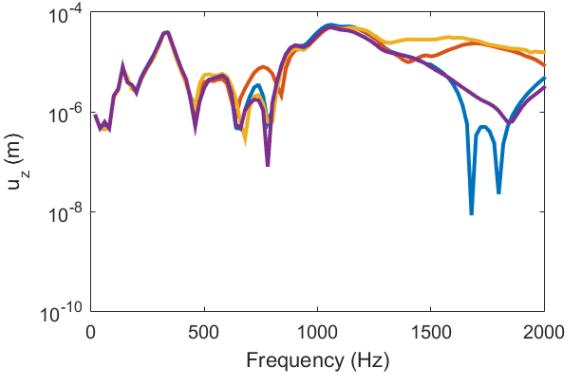
Results at probe point - $p_{fluid} = p_{solid}$

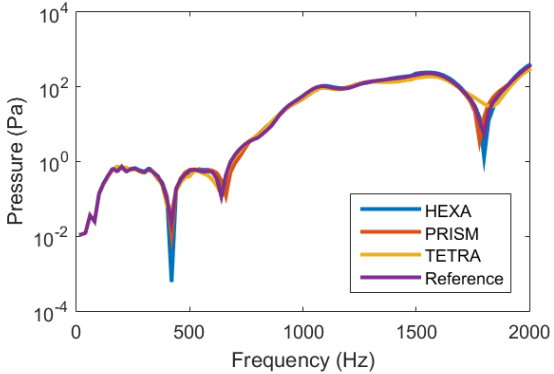
- Here we enforce $p_{fluid} = p_{solid}$
- Similar results





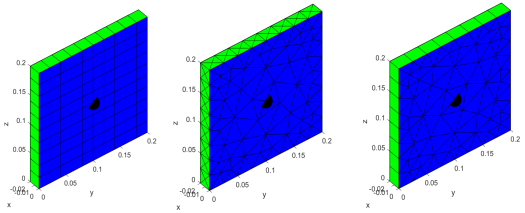


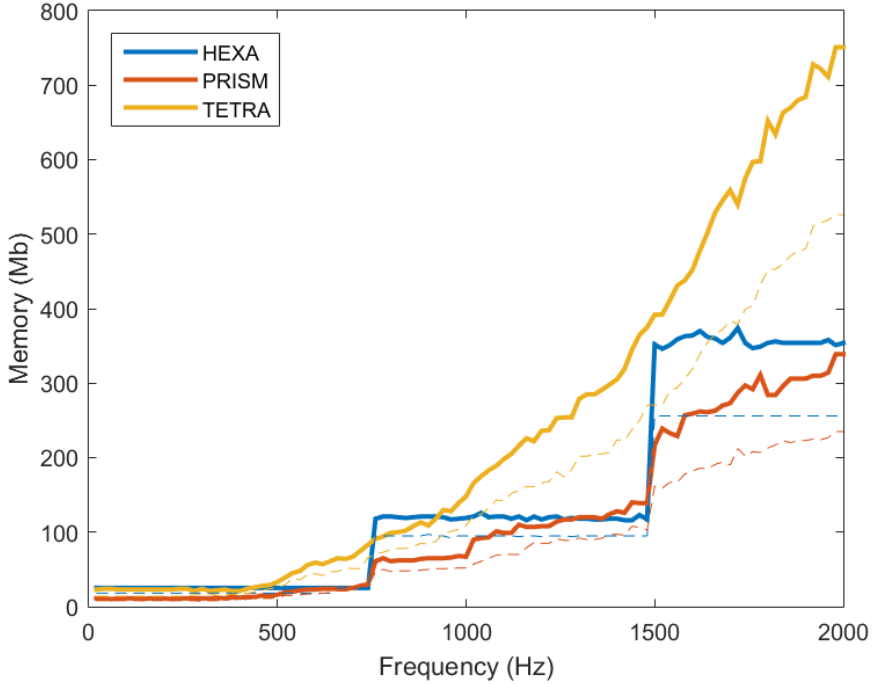




What is the gain in reducing $p{\downarrow}fluid$?

- Significant gains in using a different order for the fluid phase (~25-30% reduction in memory)
- HEXA and PRISM elements more efficient than the TETRA





6. Conclusion and future work

Conclusions

- Static condensation should be included
- No drastic cost reduction for $p>4$ but still a positive impact of order increase
- Investigated a simple error indicator to determine the order in each element
- Error is not well controlled but overall, the dynamics is still captured
- Significant gain in using a lower order for the fluid phase

Future work

- New error indicator, with a single element solving 1D Biot equations instead of 1D Helmholtz with complex wavenumber
- 3D case with analytical solution (periodic BC)
- Test other materials and other configurations (multi-layered media)