On numerical methods for the computation of guided waves involving porous materials

A travel through the waveguide

B. Nennig[*], Y. Aurégan[+], W. Bi.[+], R. Binois[*,#], N. Dauchez[#], J.-P. Groby[+], E. Perrey-Debain[#], L. Xiong[+]

[*]Institut supérieur de mécanique de Paris (SUPMECA), Laboratoire Quartz EA 7393, 3 rue Fernand Hainaut, 93407 Saint-Ouen, France.
[+]Laboratoire d'Acoustique de l'Université du Maine, Unité Mixte de Recherche CNRS 6613, Avenue Olivier Messiaen, 72085 Le Mans Cedex 9, France
[#]Sorbonne universités, Université de Technologie de Compiègne, Laboratoire Roberval, UMR CNRS 7337, CS 60319, 60203 Compiègne cedex, France.

benoit.nennig@supmeca.fr

SAPEM 2017,
December 4-6 2017, Le Mans, France

Introduction

Disclaimer !

Focuses mainly on noise control and acoustic duct silencers

Application fields

Guided waves in porous materials can be found in many noise control applications

(a) Muffler, perforated sheet and porous

(b) Engine intake, several studies to put porous

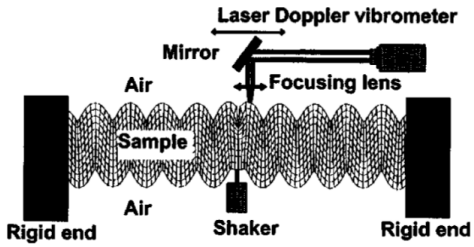
(c) Splitter silencer, alternating layer of air and glasswool

- Provide ideal channels for noise transmission (buildings, aircraft, ...)

Application fields

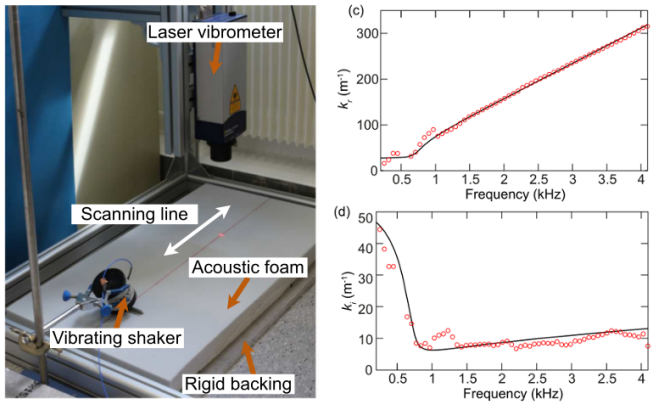
Guided waves can be useful to get information about poroelastic material skeleton.

- Direct measurement of quasi *Lamb waves*



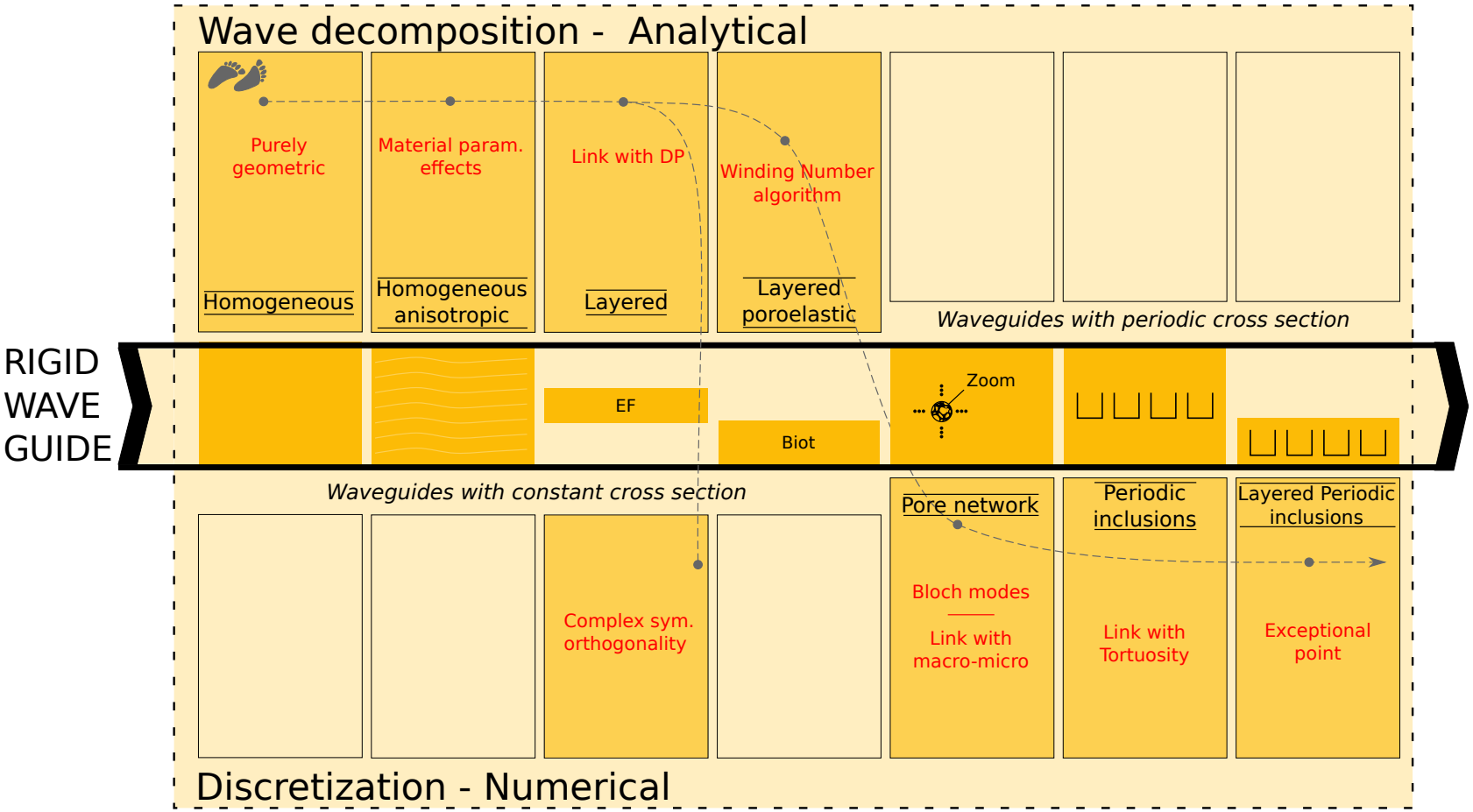
Finite element simulation of the deformation at 200 Hz of an undamped elastic plate, with the effective properties of the porous layer saturated by air under Lamb conditions (foam 1), for a line source located at the center.

(d) [Boecks, 2005]



(e) [Geslain et al., 2017] SLaTCoW method

Outline


Wave decomposition - Analytical
Purely geometric
Material param. effects
Link with DP
Winding Number algorithm
Homogeneous
Homogeneous anisotropic
Layered
Layered poroelastic
Waveguides with periodic cross section
RIGID WAVE GUIDE
EF
Biot
Zoom
Waveguides with constant cross section
Pore network
Periodic inclusions
Layered Periodic inclusions
Bloch modes
Link with macro-micro
Link with Tortuosity
Exceptional point
Complex sym. orthogonality
Discretization - Numerical

Outline

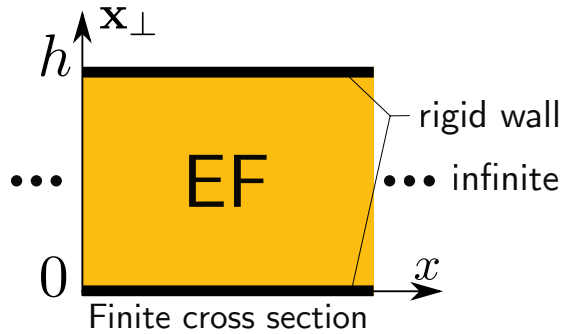
1. Introduction
   - Application fields
   - Outline
   - Assumptions
2. Waveguides with constant cross section
   - Homogeneous Waveguides
   - Transverse isotropics waveguides
   - Layered waveguides
   - Layered poroelastic waveguides
3. Waveguides with periodic cross section
   - Periodic pore modeling
   - Metaporous
4. Conclusion and perspectives
   - Conclusions
   - Persepectives in numerical methodes

Assumptions

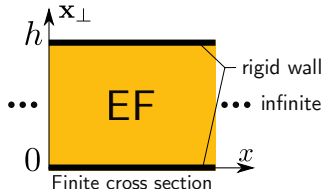
- Frequency domain $\mathrm{e}^{-\mathrm{i}\omega t}$
- Infinite duct, only few discussion on matching condition and so on
- No flow here (see Brambley Wednesday !)
- Rigid waveguide
- Weak modal density, for high modal density [Bi, 2014 ; Chan, 2017] (see Perrey-Debain Wednesday)

Waveguides with constant cross section

Homogeneous Waveguides



Homogeneous Waveguides - Problem statement



For a an infinite duct with *invariant* cross section, the pressure field can be written as $p(\mathbf{x}) = \phi(\mathbf{x}_\perp)\mathrm{e}^{\mathrm{i}\beta x}$, where, β is the axial wavenumber
If the porous material fulfills Helmholtz eq., with the wavenumber k_p

$$\Delta p + k_p^2 p = 0, \quad \text{PDE inside the waveguide}$$
$$\Delta_\perp \phi + (k_p^2 - \beta^2)\phi = 0,$$
$$\Delta_\perp \phi + \alpha^2 \phi = 0, \quad \text{Where, } \alpha \text{ is the tranverse wavenumber}$$
$$\nabla_\perp \phi \cdot \mathbf{n} = 0, \quad \text{Boundary condition.}$$

ϕ and α fulfill an eigenvalue problem. Here α are *real* and *positive*. The axial wavenumber $\beta_n = \sqrt{k_p^2 - \alpha_n^2}$ is complex, when $k_p \in \mathbb{C}$.

- The problem is purely geometric
- Cut-off freq. in real case

Homogeneous Waveguides - Resolution

Wave decomposition

- If the cross section is separable coordinate system for the wave equation, the wave propagation can solved explicitly. ex 1D : $\phi(y) = A\mathrm{e}^{\mathrm{i}\alpha y} + B\mathrm{e}^{-\mathrm{i}\alpha y}$.
- The BC (rigid) yield to a small size NL eigenvalue problem in α

$$\underbrace{\begin{bmatrix} 1 & 1 \\ \mathrm{e}^{\mathrm{i}\alpha h} & \mathrm{e}^{-\mathrm{i}\alpha h} \end{bmatrix}}_{\mathbf{M}} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  To solve it, generally we look for the root of the determinant

$$\det \mathbf{M}(\alpha) = 0.$$

- Newton-Raphson method, Muller's method, the Secant method or simplex method : **iterative algorithms requiering initial approximations** for the zeros
- Other method based on Argument Principle (see later [Nennig,2010 ; Kravanja,2000]), **without initial guess**

Homogeneous Waveguides - Resolution (cont.)

Numerical methods

- Finite element method, the discretization of the weak formulation yields to a *real symmetric generalized eigenvalue problem*

$$(\mathbf{K} - \alpha^2 \mathbf{M})\phi = \mathbf{0}$$

  leading to orthogonal eigenvectors and positive eigenvalues.
- Finite difference, similar results
- Wave finite element [Mace, 2005 ; Serra, 2015]
- Projection on orthogonal basis like Chebyshev spatial discretization technique [Chan, 2017], or the multimodal method [Bi, 2007] real symmetric matrix du to weak formulation.

▶ All first eigenvalue are obtained, without initial guess.
▶ Big size problem, sparse matrix

Homogeneous Waveguides - orthogonality

The solution ϕ_n and α_n are orthogonal. If we take 2 modes

$$\Delta_\perp \phi_n + \alpha_n^2 \phi_n = 0$$
$$\Delta_\perp \phi_m + \alpha_m^2 \phi_m = 0$$
$$+\text{rigid BC}$$

After multiplying by ϕ_m^* and integrating by part twice and using the BC, we get

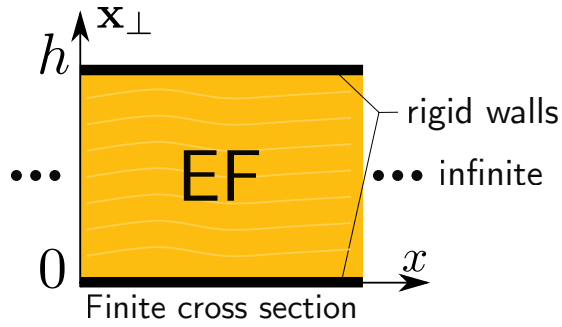
$$(\alpha_n^2 - \alpha_m^2) \underbrace{\int_S \phi_m^* \phi_n \, \mathrm{d}S}_{L^2-\text{inner product}} = 0.$$

This, the orthogonality relation reads

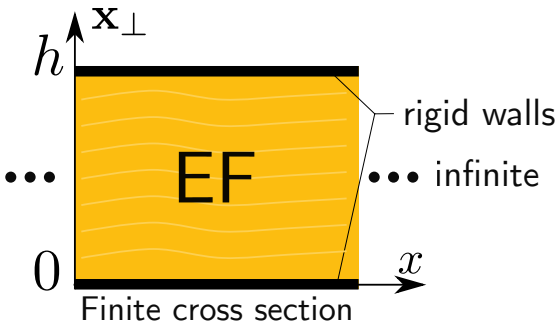
$$\int_S \phi_m^* \phi_n \, \mathrm{d}S = \Lambda_n \delta_{mn}.$$

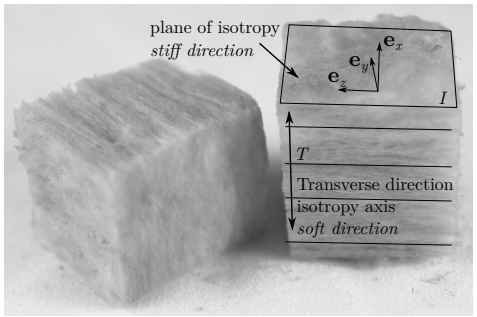
- Changing boundary condition may change everything (ex : impedance)

Transverse isotropics waveguides



Transverse isotropics waveguides





Consider a 2D rigid-walled waveguide of height h filled with a TI fluid [Nennig et al., submitted JASA].
The wave propagation is described by [De Hoop, 1995 ; Torrent, 2008 ; Norris, 2009]

$$\nabla \cdot (\boldsymbol{\tau} \nabla p) + \frac{\omega^2}{K_{\mathrm{eq}}} p = 0,$$

where $\boldsymbol{\tau} = \boldsymbol{\rho}^{-1}$ is the inevrse of the density tensor.

We are looking for a guided mode propagating along the $\mathbf{e}_x$ direction, having the exponential form $p = \phi(y)\mathrm{e}^{\mathrm{i}\beta x}$ where

$$\phi(y) = a\mathrm{e}^{\mathrm{i}\alpha_1 y} + b\mathrm{e}^{\mathrm{i}\alpha_2 y}.$$

Transverse isotropics waveguides (cont.)

After insertion into the wave equation, transverse wavenumbers α_1 and α_2 must satisfy

$$\tau_{yy}\alpha_i^2 + 2\beta\tau_{xy}\alpha_i + \tau_{xx}\beta^2 - \frac{\omega^2}{K_{\text{eq}}} = 0, \quad \text{with } i = 1, 2.$$

Applying rigid conditions at the wall, i.e. $\mathbf{u}^w \cdot \mathbf{e}_y = 0$, leads to a boundary condition of mixed type and non-trivial solutions exist if

$$\sin\left(\frac{h\sqrt{\Delta}}{2\tau_{yy}}\right) = 0$$

where

$$\Delta = 4\tau_{yy}\frac{\omega^2}{K_{\text{eq}}} + 4\beta^2(\tau_{xy}^2 - \tau_{xx}\tau_{yy})$$

is the discriminant of the quadratic equation.

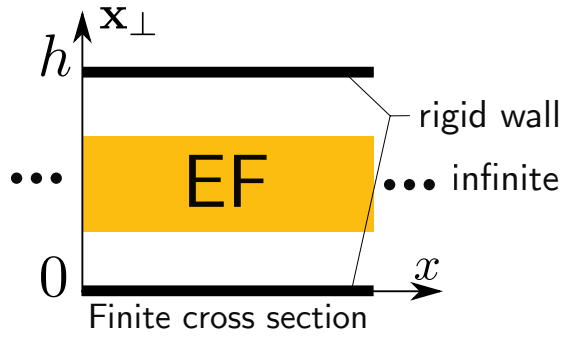
Transverse isotropics waveguides (cont.)

Propagation constant for guided modes are then found explicitly as

$$\beta_n = \pm \frac{\tau_{yy}}{\sqrt{\tau_T \tau_l}} \sqrt{\frac{\omega^2}{\tau_{yy} K_{\mathrm{eq}}} - \left(\frac{n\pi}{h}\right)^2}$$

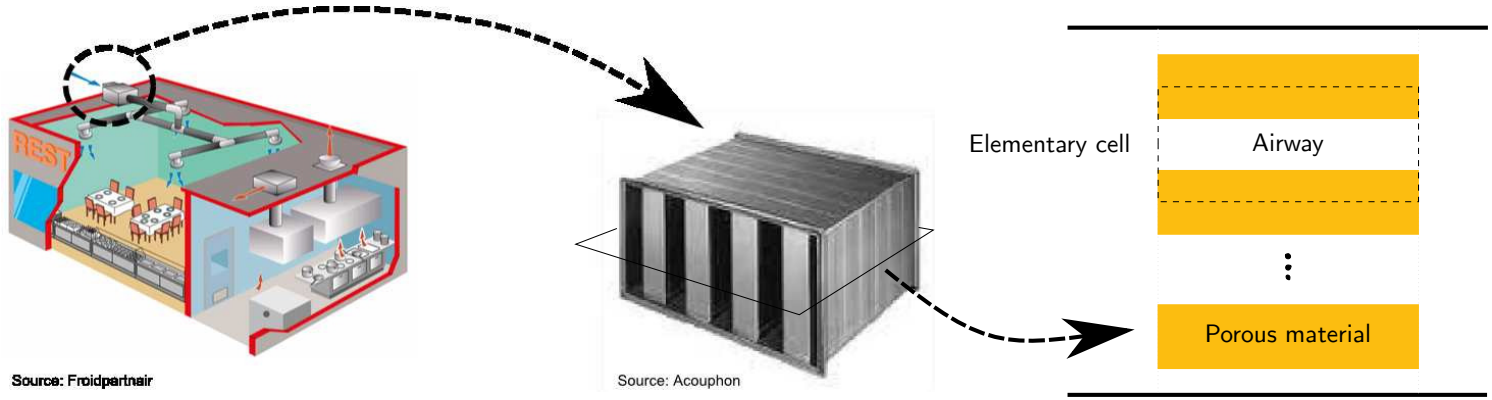
- Right and left propagating modes are identical
- The fundamental mode, $n = 0$, is also function of the transverse coordinate y except when $\Theta = 0$ or $\pi/2$
- Solution depends on the fluid properties, not only on the WG shape

Layered waveguides



Layered waveguides

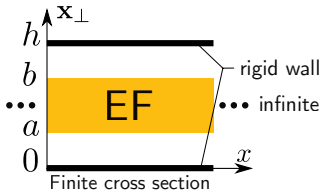
HVAC Silencers are most often made of parallel baffles of porous material (rockwool) well described by equivalent fluid model



- Weak attenuation at low frequency [63 Hz - 500 Hz].
- For this frequency range fundamental mode dominates
- Fundamental mode range depends on both the duct height h and of the number of baffle N : $n = n^{inc} + 2qN$. [Mechel, 1990]

Refs : [Tam et al., 1991 ; Aurégan et al., 2001 ; Kirby et al., 2005 ; Binois et al., 2015 ; Nennig et al., 2015]

Layered waveguides



In layered waveguide, each layer $i = 1, I$, satisfies

- The Hemlholtz equation, with different materials properties like density ρ_i, celerity c_i and wavenumber k_i

$$\Delta p + k_i^2 p = 0$$

- Continuity of the pressure at each interfaces
- Continuity of the velocity ie $\left[\frac{1}{\rho_i}\nabla_\perp p \cdot \mathbf{n}\right] = 0$ at each interfaces

Applying wave decomposition and continuity condition yields

$$f(\beta) = \sum_{i=1}^{I} \frac{\alpha_i}{\rho_i} \tan \alpha_i h_i$$

where $\alpha_i = \sqrt{k_i^2 - \beta^2}$. This equation must solved in β, the solution depends on media properties.

Layered waveguides

The associated weak formulation, is obtained by multiplying the wave equation (12) by a test function ψ and by integrating over the porous media occupying an arbitrary domain Ω of boundary S

$$\sum_{i=1}^{I} \frac{1}{\rho_i} \left(- \int_\Omega \nabla\psi \cdot \nabla\phi \, \mathrm{d}\Omega + \left(k_i^2 - \beta^2\right) \int_\Omega \psi\phi \, \mathrm{d}\Omega \right) = \underbrace{- \sum_{i=1}^{I} \frac{1}{\rho_i} \int_S \psi \frac{\partial \phi}{\partial n} \, \mathrm{d}S}_{=0}.$$

All boundary term vanished due rigid duct and transverse velocity continuity.
This yields *complex symetric eigenvalue problem*

$$\left(\begin{bmatrix} \mathbf{K}_1 + k_1^2 \mathbf{M}_1 & \dots & 0 \\ \vdots & \bigoplus & \vdots \\ 0 & \dots & \mathbf{K}_I + k_I^2 \mathbf{M}_I \end{bmatrix} - \beta^2 \begin{bmatrix} \mathbf{M}_1 & \dots & 0 \\ \vdots & \bigoplus & \vdots \\ 0 & \dots & \mathbf{M}_I \end{bmatrix} \right) \begin{bmatrix} \boldsymbol{\psi}_1 \\ \vdots \\ \boldsymbol{\psi}_I \end{bmatrix} = \mathbf{0}$$

- Complex symmetric eigenvalue problem

$\bigoplus$ stand for FEM assembly

Layered waveguides

Complex symmetric problems

Properties

- Loss of orthogonality for L^2 scalar product $\int_S \phi_n^* \phi_m \, \mathrm{d}S = 0$
- Left and right eigenvectors are still orthogonal (adjoint), bi-orthogonality

▶ For complex symmetric problem (excepted for double root), a new orthogonality relation can be obtained without conjugate (not scalar product !)

$$\int_S \phi_m \phi_n \, \mathrm{d}S = \Lambda_n \delta_{mn}.$$

eg : Impedance lining, layered materials, . . .
Refs : [Lawrie, 1999, 2006 ; Redon, 2011 ; Moiseyev, 2011]

▶ The EP/double root case will be discussed later

Layered waveguides - Example splitter silencer

With one layer of each material (air, porous), approximate solution can be obtained.

$$\frac{\alpha_p}{\rho_p}\tan\alpha_p b + \frac{\alpha_a}{\rho_a}\tan\alpha_a a = 0,$$

with $\alpha_p = \sqrt{k_p^2 - \beta^2}$ and $\alpha_a = \sqrt{k_a^2 - \beta^2}$.

- At low frequencies (viscous regime), the wavenumber order of magnitude in the porous material $k_p \propto \sqrt{\omega}$, is larger than the axial wavenumber $|\beta|$.
- When $|k_m| \gg |\beta|$, the equation (19) is equivalent to a waveguide lined with a locally reacting material satisfying

$$\frac{\mathrm{i}k_a}{Z} + \frac{\alpha_a}{Z_a}\tan\alpha_a a = 0,$$

with $Z_a = \rho_a c_0$ and $Z = \mathrm{i}\rho_p c_p \cot k_p b$.

This is equivalent to say that only plane wave propagate in the transverse direction $\mathrm{e}^{\pm \mathrm{i}k_p y}$. Applying the continuity of the pressure between the micro-porous p_m and the airway p_a, yields to

$$p_p(y) = \frac{p_a}{2\cos k_p b}\left(\mathrm{e}^{\mathrm{i}k_p y} + \mathrm{e}^{-\mathrm{i}k_p y}\right), \forall\, y \in [-b, b].$$

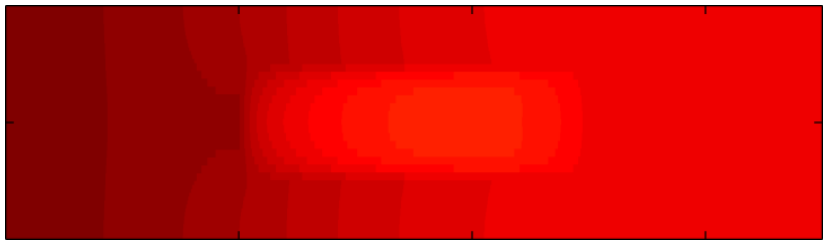
Layered waveguides - Example splitter silencer (cont.)

The pressure in the airway may considered as a constant in the transverse direction and is a forcing term for the pressure in the micro-porous media.

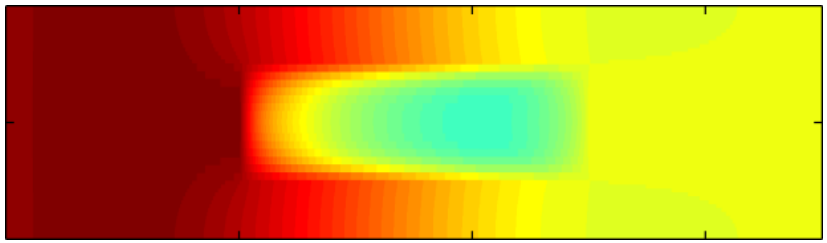
With these assumptions, the average pressure over the elementary cell reads,

$$\frac{\langle p_p \rangle}{p_a} = \frac{1}{2b} \int_{-b}^{b} p_p(y)\,\mathrm{d}y = \frac{\tan k_p b}{k_p b}.$$

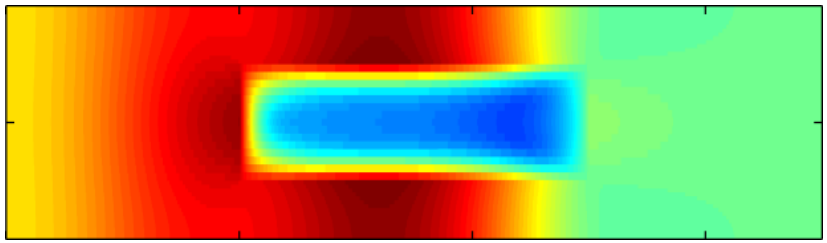
is similar to the pressure diffusion function F_d of the double porosity model [Olny et al., 2003].



(a) $f = \frac{f_d}{3}$



(b) $f = f_d$

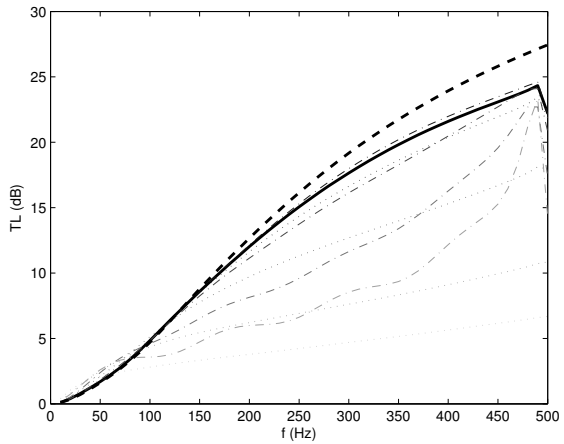


(c) $f = 3f_d$

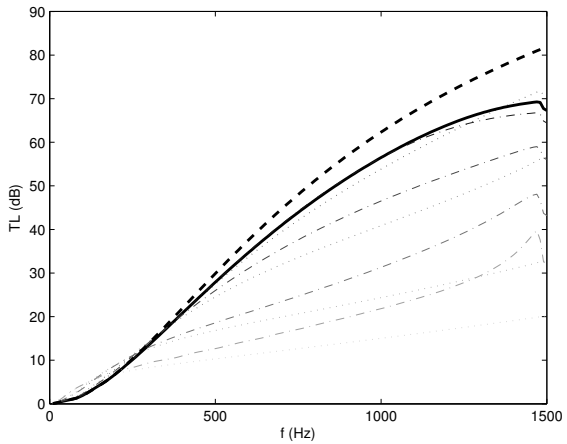
Normalized pressure field (in absolute value) below, around and above ω_d for a rockwool.

Layered waveguides - Optimal design

This closed form solution yields an optimal value ($\max \Im\mathfrak{m}\, \beta$) for the airflow resitivity $\sigma_0^* = \sqrt{\frac{3\rho_0 P_0}{b^2\phi^2}\left(\frac{1}{\gamma\alpha_\infty} + \frac{\bar{\phi}_p\phi}{\phi_p\alpha_\infty} + \Upsilon\right)}$ depending on the waveguide height h and open area ratio ϕ_p.



(a) $\phi_p = 0.5$, $N = 1$

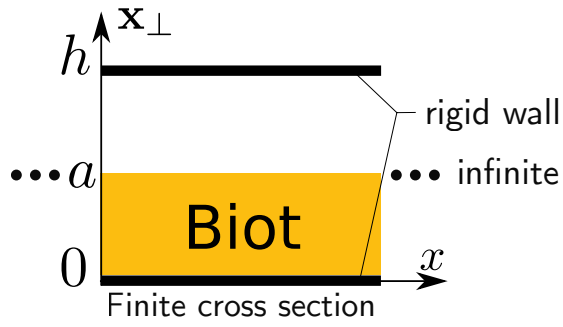


(b) $\phi_p = 0.5$, $N = 3$

Transmission loss $\sigma \in [0.1\sigma_0^*, \ldots, 10\sigma_0^*]$. Obtained with a reference solution [Binois, 2013] on realistic silencers. If $\sigma < \sigma_0^*$, (· · ·), If $\sigma > \sigma_0^*$,(— · —), If $\sigma = \sigma_0^*$, (———). Geometry $H = 0.70$ m and $L_s = 1.20$ m.

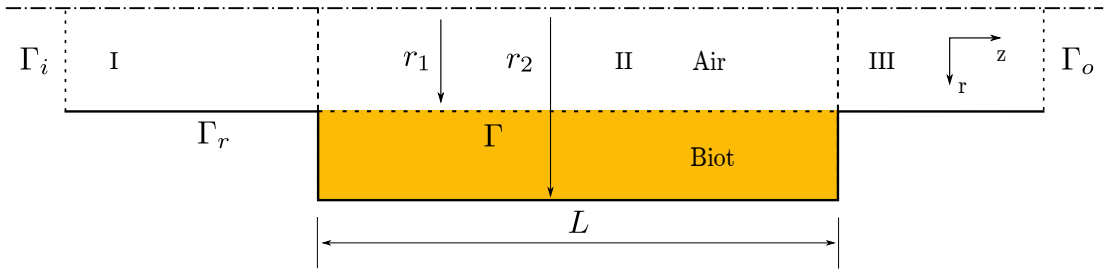
- What happens for absorption at normal incidence $\sigma = \sigma_0^*$?

Layered poroelastic waveguides



Layered poroelastic waveguides

Typical dissipative silencers (automotive industry, HVAC systems,...) consists of a cylindrical expansion chamber filled with a sound absorbing material.



- Prediction of the Transmission Loss (TL) is usually made using Mode Matching Methods (Kirby & Denia, 2007 ; Lawrie & Kirby, 2005 ; and plenty more...)
- They all consider "fluid equivalent models" (Beraneck, 1947 ; Zwikker & Kosten, 1949 ; Delany & Bazley, 1970)

▶ Why take into account the frame elasticity ? Can the skeleton improve the TL ?

Constitutive equations

For time-harmonic representation, the pressure p in the airway obeys to the (convected) wave equation

$$\Delta p - \frac{1}{c_0^2}\frac{\mathrm{d}^2 p}{\mathrm{d}t^2} = 0$$

In the poroelastic material [Biot, 1956]

$$\begin{aligned} \nabla \cdot \boldsymbol{\sigma}^s &+ \omega^2(\rho_{11}\mathbf{u} + \rho_{12}\mathbf{U}) = 0, \\ \nabla \cdot \boldsymbol{\sigma}^f &+ \omega^2(\rho_{12}\mathbf{u} + \rho_{22}\mathbf{U}) = 0, \end{aligned}$$

where $\mathbf{U}$ et $\mathbf{u}$ are respectively the fluid and the solid phase displacements and the Solid and fluid phase stress tensors are given by

$$\begin{aligned} \boldsymbol{\sigma}^s &= (A\,\nabla \cdot \mathbf{u} + Q\,\nabla \cdot \mathbf{U})\,\mathbf{I} + 2N\,\boldsymbol{\varepsilon}^s \\ \boldsymbol{\sigma}^f &= (Q\,\nabla \cdot \mathbf{u} + R\,\nabla \cdot \mathbf{U})\,\mathbf{I} \end{aligned}$$

where $\boldsymbol{\varepsilon}^s = 1/2(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}})$ is the usual strain tensor.

Helmholtz decomposition

In the airway, we look for purely acoustic mode :

$$\mathbf{w} = \nabla \varphi_0$$

In the porous media, both displacement fields are written as

$$\mathbf{u} = \nabla \varphi + \nabla \wedge \boldsymbol{\psi} \quad \text{and} \quad \mathbf{U} = \nabla \chi + \nabla \wedge \boldsymbol{\Theta}.$$

After equations decoupling, we have

$$\varphi = \varphi_1 + \varphi_2 \ , \quad \chi = \mu_1 \varphi_1 + \mu_2 \varphi_2 \ , \quad \text{and} \quad \boldsymbol{\Theta} = \mu_3 \boldsymbol{\psi}$$

All potentials fulfill wave equation. Wave number k_i and phase coupling coefficients μ_i are known (Allard, 1993)

Helmholtz decomposition > General potential form

We put

$$X(r,\theta,z,t) = \tilde{X}(r,\theta)\,\mathrm{e}^{\mathrm{i}(\beta z-\omega t)}$$

Following (Gazis, 1951) we find for symetric modes

$$\begin{aligned}
\tilde{\varphi}_1 &= [A_1 J_m(\alpha_1 r) + B_1 Y_m(\alpha_1 r)]\cos m\theta \\
\tilde{\varphi}_2 &= [A_2 J_m(\alpha_2 r) + B_2 Y_m(\alpha_2 r)]\cos m\theta \\
\tilde{\psi}_r &= [A_3 J_{m+1}(\alpha_3 r) + B_3 Y_{m+1}(\alpha_3 r)]\sin m\theta \\
\tilde{\psi}_\theta &= -[A_3 J_{m+1}(\alpha_3 r) + B_3 Y_{m+1}(\alpha_3 r)]\cos m\theta \\
\tilde{\psi}_z &= \left[A_3' J_m(\alpha_3 r) + B_3' Y_m(\alpha_3 r)\right]\sin m\theta
\end{aligned}$$

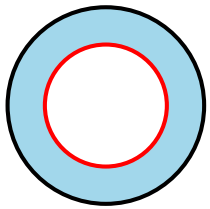
Similarly

$$\tilde{\varphi}_0 = A_0 J_m(\alpha_0 r)\cos m\theta$$

Continuity conditions

At the air-porous interface, the following coupling conditions must be verified

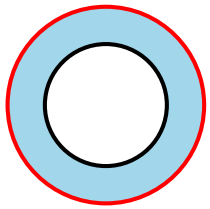
$$\begin{aligned} \boldsymbol{\sigma}^t \mathbf{n} &= -p\,\mathbf{n} \\ p_p &= p \\ \underbrace{\phi \mathbf{U}\cdot\mathbf{n} + (1-\phi)\mathbf{u}\cdot\mathbf{n}}_{\text{Porous side}} &= \underbrace{\mathbf{w}\cdot\mathbf{n}}_{\text{Airway}} \end{aligned}$$



Here ϕ is the porosity and the pore pressure p_p is obtained from the fluid phase tensor as $-3\phi p_p = \mathrm{tr}\boldsymbol{\sigma}^f$.

On the hard surface the poroelastic layer is assumed to be **clamped**

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad \mathbf{U}\cdot\mathbf{n} = 0.$$



Continuity conditions > Eigenvalue equation

The modal vector containing the wave potential amplitudes

$$\mathbf{V} = \left(A_0, A_1, B_1, A_2, B_2, A_3, B_3, A_3', B_3'\right)^{\mathrm{T}},$$

must be a non-trivial solution of the 9×9 algebraic system after boundary conditions application

$$\mathrm{M}(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta)\mathbf{V} = \mathbf{0}$$

Together with the dispersion equation (once inverted)

$$\alpha_0 = \sqrt{(k_0 - M\beta)^2 - \beta^2} \quad \text{and} \quad \alpha_i = \sqrt{k_i^2 - \beta^2}, \quad i = 1, 2, 3$$

we have

$$\mathrm{M}(\beta)\mathbf{V} = \mathbf{0}$$

Layered poroelastic waveguides

Instead of finding roots of $f(\beta) = \det \mathrm{M}(\beta)$ it's possible to find the zeros of the associated polynomial Π with the same zeros [Kravanja & Van Barel, 2000 ; Nennig et al., 2010]

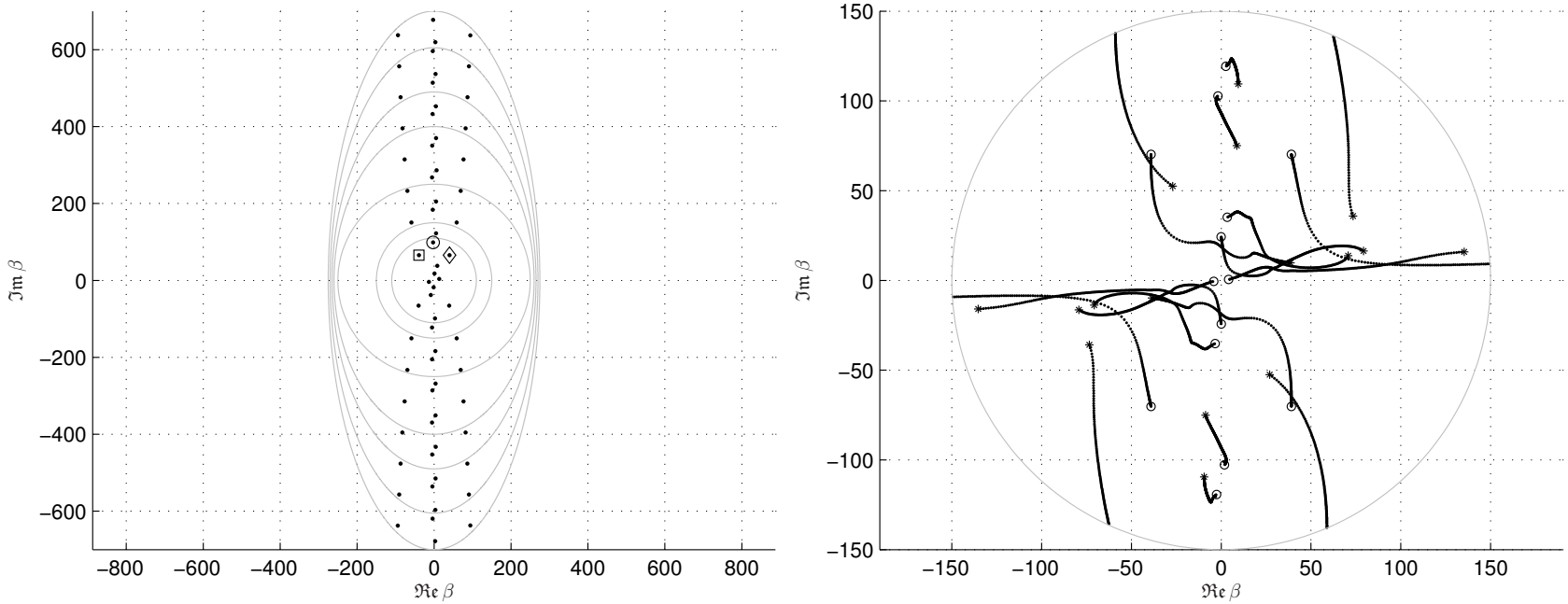
1. Residue theorem gives for analytical functions

$$S_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{f'(\beta)}{f(\beta)} \beta^n \, \mathrm{d}\beta = \sum_{k=1}^{N_\beta} n_k \beta_k^n$$

2. The coefficients of the polynom Π are related to S_n
3. Get the zeros of the polynom Π by computing the eigenvalues of companion matrix ie the root of $f(\beta)$ in the interior of C

- Important to use integer number of sample on C
- Works with all closed waveguides (always analytics function)
- Variant with log, there is efficient way to generalized this to big matrix [Spence, 2004 ; Güttel, 2017]...

Roots in the complex plane > frequency analysis



(a) Integration path

(b) Frequency sweep

Roots in β complex plan for XFM foam in the silencer A and $M = 0$.

- Good numerical satiblity, because it never looks close to the root

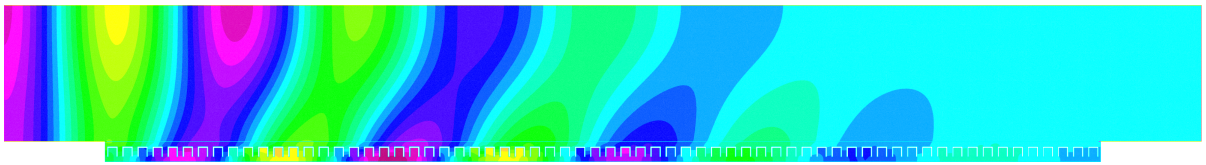
Waveguides with periodic cross section

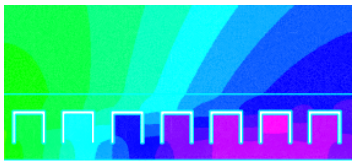
As the governing equations, the boundary conditions, and the geometry are **d**-periodic, it follows from the *Bloch theorem* that the solutions are Bloch waves,

$$\phi(\mathbf{x}) = \hat{\phi}(\mathbf{x})\mathrm{e}^{\mathrm{i}\mathbf{k}_B\cdot\mathbf{x}}, \tag{1}$$

i.e. the fields can be split into a **d**-periodic field $\hat{\phi}(\mathbf{x})$ modulated by a plane wave involving the Bloch wavevector $\mathbf{k}_B = k_B\boldsymbol{\kappa}$, where the unit vector $\boldsymbol{\kappa} = \mathbf{k}_B/k_B$ stands for the propagation direction.

- The real part of $\mathbf{k}_B$ measures the change in phase across the cell
- The imaginary part of $\mathbf{k}_B$ measures the attenuation.



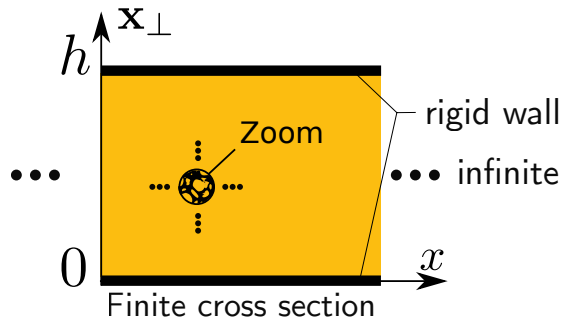


Two way for solving this [Collet, 2011 ; Frazier, 2017]

- ✗ Choose ω as eigenvalue, suitable for free response, difficult with parameters $f(\omega)$
- ✓ Choose k_B as eigenvalue, suitable for forced response, i. e. most application in waveguides

▶ Semi-analytical methods using plane wave expansion are also possible [Duclos, 2009 ; Deymier, 2013]

Periodic pore modeling



Visco-thermal fluid

We use here *linearized Navier and Stokes equations* with Fourier flux $\mathbf{q} = -\kappa \nabla T$, and we assume a Newtonian fluid, ideal gas [Stinson, 1991]

$$
\begin{aligned}
-\mathrm{i}\omega\rho_0 \mathbf{v} + \nabla p - \nabla \cdot \boldsymbol{\tau} &= \mathbf{0}, \\
-\mathrm{i}\omega\rho_0 C_p T - \kappa \Delta T + \mathrm{i}\omega p &= Q, \\
\nabla \cdot \mathbf{v} + \mathrm{i}\omega \frac{T}{T_0} - \mathrm{i}\omega \frac{p}{P_0} &= 0.
\end{aligned}
$$

Where $\mathbf{v}$ is the velocity, T the temperature and p the pressure fluctuation.

Taylor-Hood element are used to avoid looking problem in Stokes like equation. The velocity and the temperature is approximated with the quadratic lagrangian element, and the pressure with linear lagrangian element [Kampinga, 2010].

- The pressure is kept to ensure numerical stability
- Possible to add the coupling with the skeleton
- Possible to add other physics

Implementation

The periodic formulation can be obtained thanks to a systematic approach, for instance

$$\nabla \cdot \mathbf{u} = [\nabla \cdot \hat{\mathbf{u}} + \mathrm{i}\mathbf{k}_B \cdot \hat{\mathbf{u}}]\, \mathrm{e}^{\mathrm{i}\mathbf{k}_B \cdot \hat{\mathbf{u}}}$$

$$\boldsymbol{\nabla}\mathbf{u} = \left[\boldsymbol{\nabla}\hat{\mathbf{u}} + \mathrm{i}\hat{\mathbf{u}}\,\mathbf{k}_B^{\mathrm{t}}\right] \mathrm{e}^{\mathrm{i}\mathbf{k}_B \cdot \mathbf{x}}$$

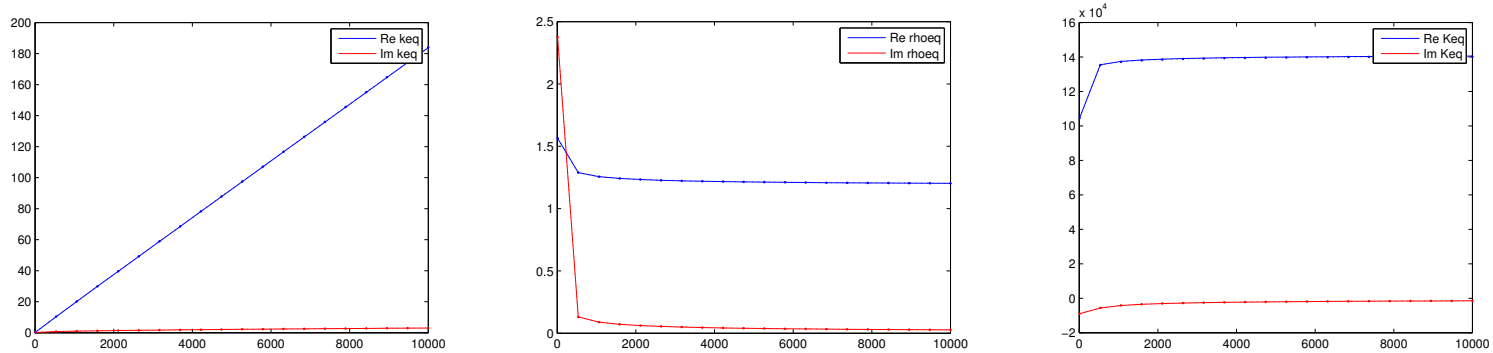
We get a *Quadratic eigenvalue problem* in k_B [Tisseur, 2001],

$$k_B^2 \mathbf{K}_2 + k_B \mathbf{K}_1 + \mathbf{K}_0 = 0$$

- $2N$ eigenvalues, only N independent eigenvectors
- Here all parameters are real, then $\mathbf{K}_2$, $\mathbf{K}_0$ real symmetric, $\mathbf{K}_1$ Hermitian (empty diag.) (similar gyroscopic pb when $\lambda = \mathrm{i}k_b$)
- k_B and $-k_B^*$ are solution
- The left eigenvector for k_{Bn} is also the right eigenvector for $-k_{Bn}^*$

▶ There is several possibilities for resolution (linearization be careful to respect symetries, direct [TOAR, SLEPc])

Validation



(a) k_{eq} (b) ρ_{eq} (c) K_{eq}

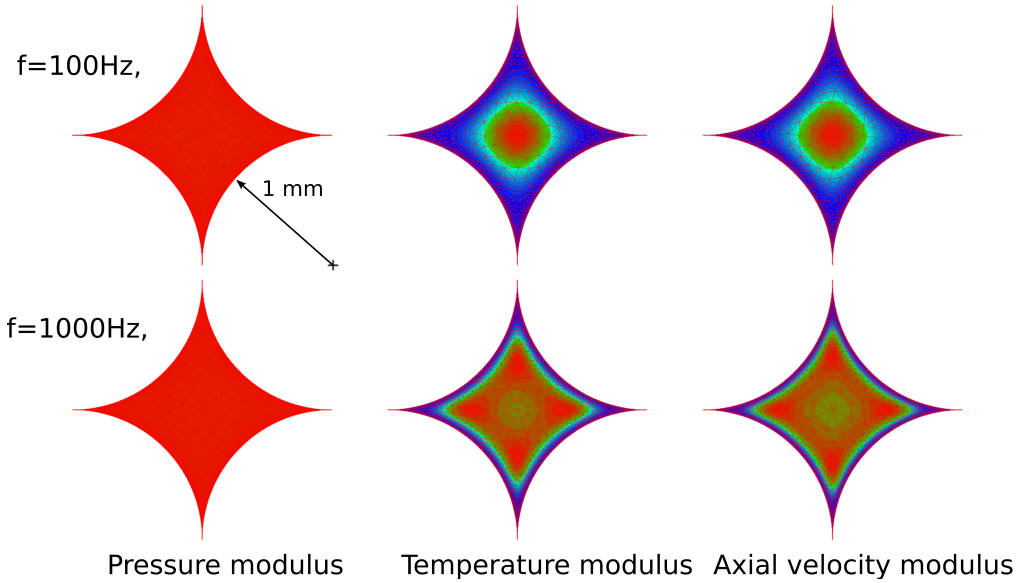
Comparaison with FEM (dot) Zwicker and Kosten solution (solid) for circular pore $r = 1$ mm.

Using the 1st eigenvector, effective parameters ρ_{eq} and K_{eq}, can be recover as defined in [Allard, 1993, Chap. 4]

$$-\mathrm{i}\omega\rho_{eq}\bar{v}_z = -\overline{\frac{\partial p}{\partial z}}. \quad \text{et} \quad K_{eq} = -\bar{p}/\overline{\nabla\cdot\left(\frac{\mathbf{v}}{-\mathrm{i}\omega}\right)}.$$

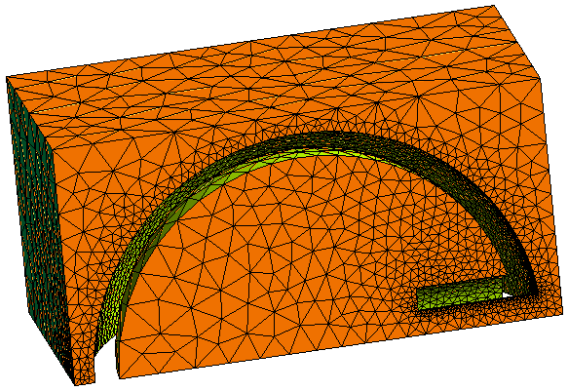
Here $\bar{}$ is the average over the cross section.

Examples

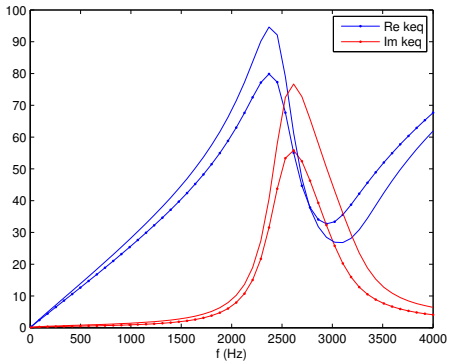


FEM computation on a staight pore with complex cross section

Periodic pore modeling



(a) géométrie

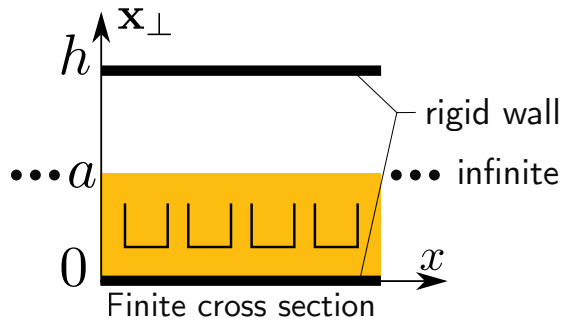


(b) k_B

Comparaison between FEM 3D model (dot) and homogenization [Boutin, 2015] (solid line) with a rigid Helmholtz resonator in CC crystal ($a = 6$ mm, $t = 0.3$ mm, $l = 1.5$ mm, $r_c = 0.3$ mm, $r_s = 2.75$ mm)

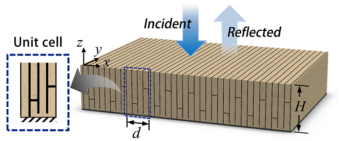
- Strong connexion with homogenization theory

Metaporous

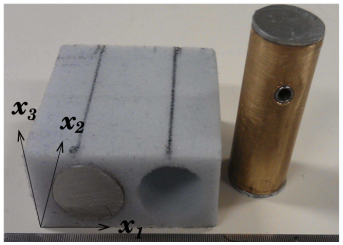

$\mathbf{x}_\perp$
h
a
0
x
rigid wall
infinite
Finite cross section

Aim

Few metaporous examples use to increase absorption, limit transmission through wall or through duct



(a) Yang et al., 2015



(b) Groby et al., 2014

(c) Lagarrigue et al., 2015

(d) Aurégan et al., 2016

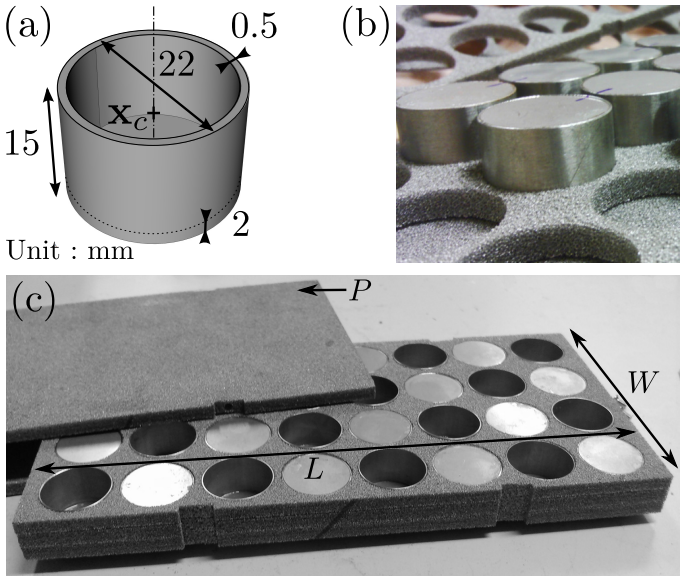
(e) Xiong et al., 2017

Focus here on duct parietal acoustic treatments

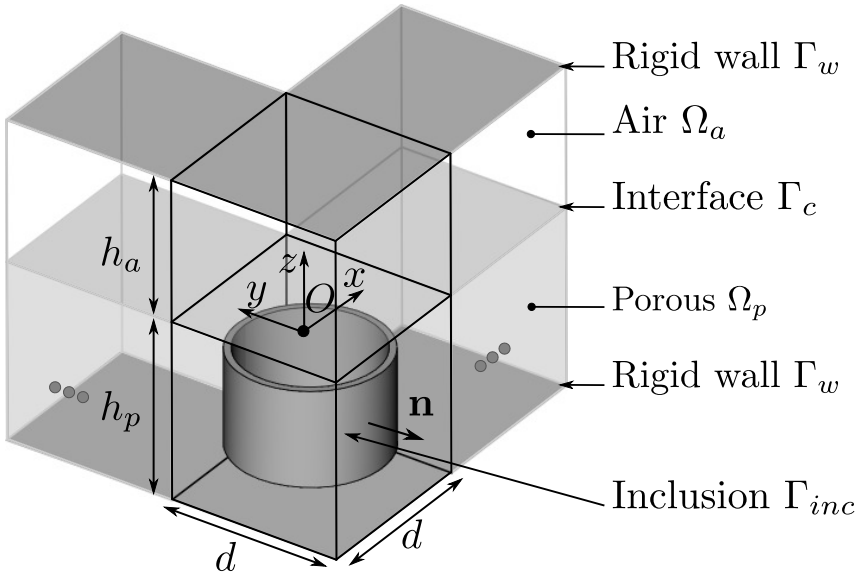
- Bloch modes yields *non hermitian quadratic eigenvalue problem*
- Geometric parameters in metamaterial are easy to tune
- Exploit *easy tuning* of metamaterial to find EP / force Bloch modes to coalesce

Configurations

Sample made of 5 mm metalic foam layers and blind cylinder inclusions with 2 possible orientation [Xiong, 2017]



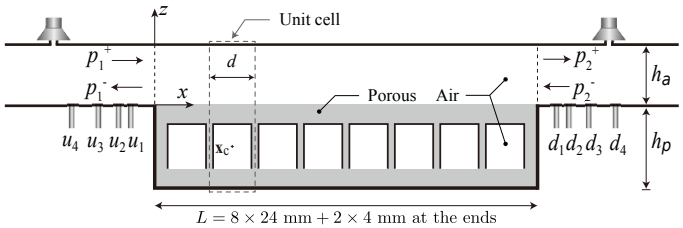
(f) Sample



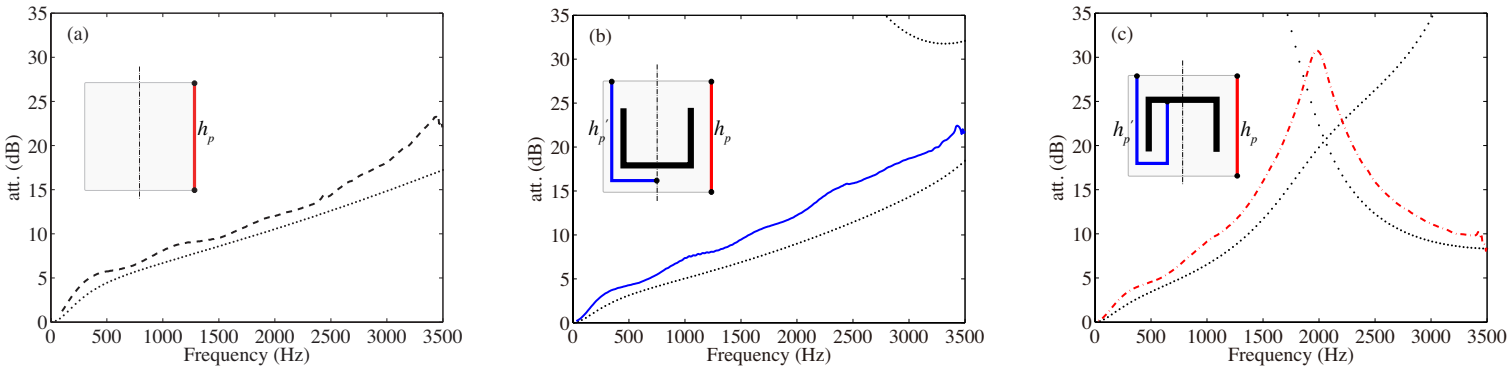
(g) Model

Pictures of (a) an open cylinder inclusion (filled with air), (b) zoom of the drilled metallic foam layer of height 5 mm, and (c) a whole sample with cylinders (Fig. 7(a)) embedded in an alternated way.

Experimental observation



A 2D schematic view of the experimental setup. The measured sample, of length $L = 200$ mm, contains 8 unit cells. $\mathbf{x}_c$ is the center of the rigid inclusion.

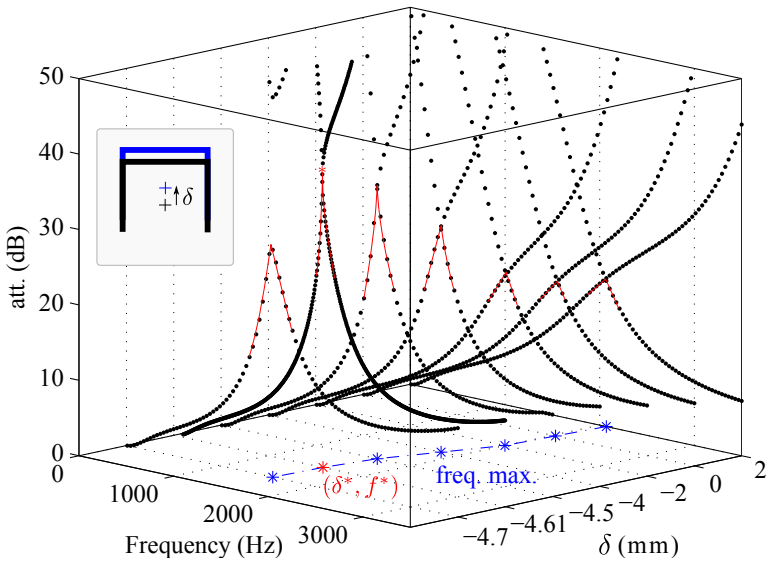


Comparison between the Bloch wave attenuation ($\cdot$) and the measured transmission loss (line) with 8 unit cells for the configurations : (a) 5P, (b) P-⊔-P, and (c) P-⊓-P.

Parametric study

The inclusion is moved from δ from the centre of the cell is positive.

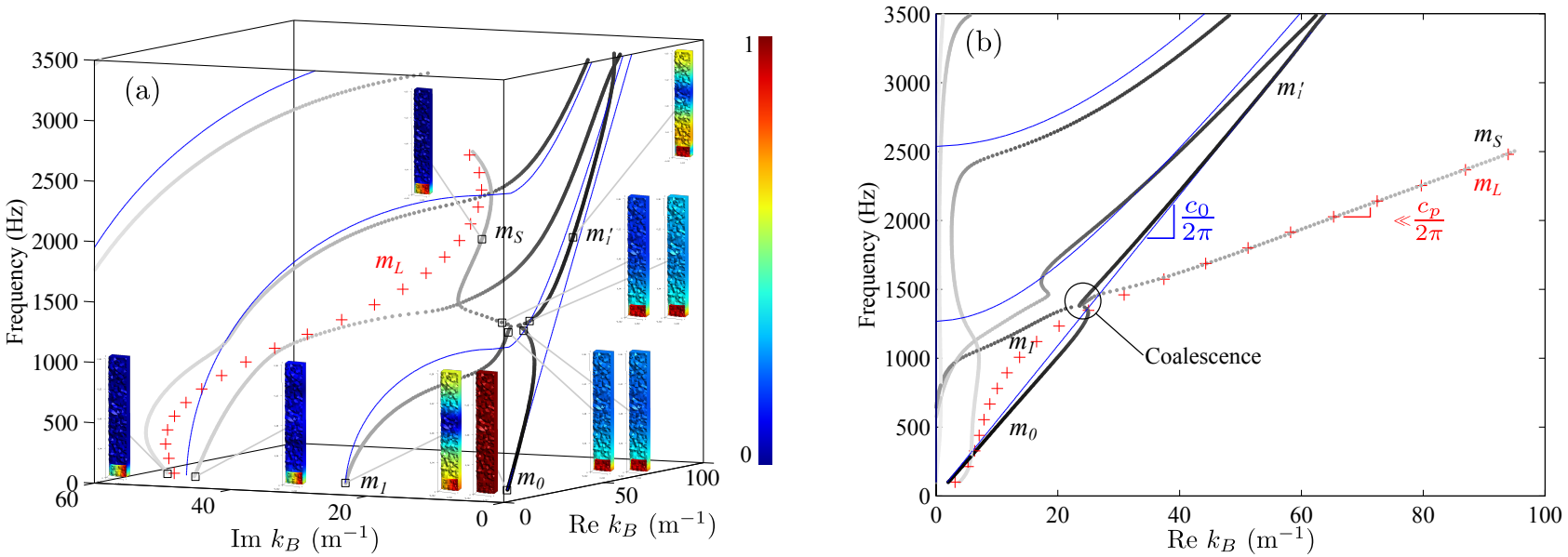
- parametric (δ, f) Quadratic eigenvalue problem



Mode attenuations for the lower two Bloch modes as a function of frequency at different inclusion positions δ.

- The attenuation reach a max. when 2 modes coalesce (eigenvector and eigenvalue)

Mode coalescence



Dispersion curves for the Bloch wavenumber when $h_a = 135$ mm and $\delta = -4.0$ mm a) 3D view, b) projection in the frequency-$\mathfrak{Re}\, k_B$ plane. The inserted pictures show the (O, x, z)-cut of the modulus of the right eigenvectors at 100, 1375, 1385 and 2000 Hz.

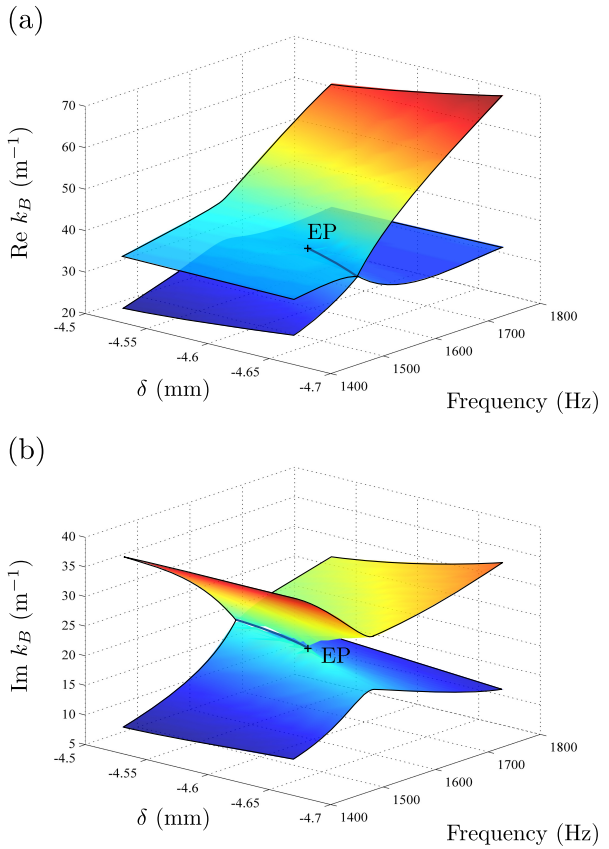
- At the coalescence the double mode is localized in the liner, similar to impedance case [Bi, 2016]
- (δ^*, f^*) is an exceptional point EP [Kato, 1980 ; Heiss,1990 ; Berry,2004 ; Bi, 2015]

Parametric study

Using a Taylor expansion in the vicinity of the double root k_B^* [Tester,1973 ; Shenderov,2000],

$$k_B - k_B^* \approx \pm \left[\left(-\frac{1}{2} \frac{\partial^2 D^*}{\partial k_B^2} \right)^{-1} \left((\delta - \delta^*) \frac{\partial D^*}{\partial \delta} + (f - f^*) \frac{\partial D^*}{\partial f} \right) \right]^{1/2}$$

▶ The EP is a branch point in the parameter plane



Riemann sheet of the real (a) and imaginary (b) part of the lower attenuated Bloch wavenumber. The line ——— indicates the crossing of the different sheets.

Conclusion and perspectives

Conclusion and perspectives

In a waveguide maximizing attenuation is different that for absorbing panel

- Partial filling / airway
- Higher modes

Recently, we find new design rules

- Tune on EP with metaporous [Xiong et al., 2017]
- Use σ_0^* when porous remains in viscous regime [Nennig et al., 2015]
- Use transverse isotropic material

Other strategies have to be tested

- Play with path in parameter space for asymmetric propagation [Doppler, 2016]

▶ Take advantage of complex problem to reveals new physics (EP, CPA, ...)
▶ Fine modeling of damping and radiation mechanism opens new routes (QM, EM)

Conclusion and perspectives

Numerical methods are now mature to tackle such problem, however, it remains some challenges or some possible improvements

- How to find efficiently EP (Jordan decomposition ?)
- How to speed-up ferquency sweep
- Use waveguide as an homogenization method
- ...

The waveguide journey comes to an end,

Thank you for your attention !

Thank you for your attention !

See you in porous session of CFA in Le Havre

This work was partly funded by the ANR Project METAUDIBLE No. ANR-13-BS09-0003-01 funded jointly by ANR and FRAE.