Boundary Conditions modeling Acoustic Linings in Flow

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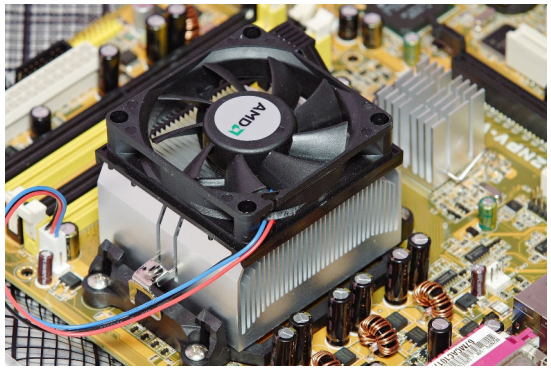
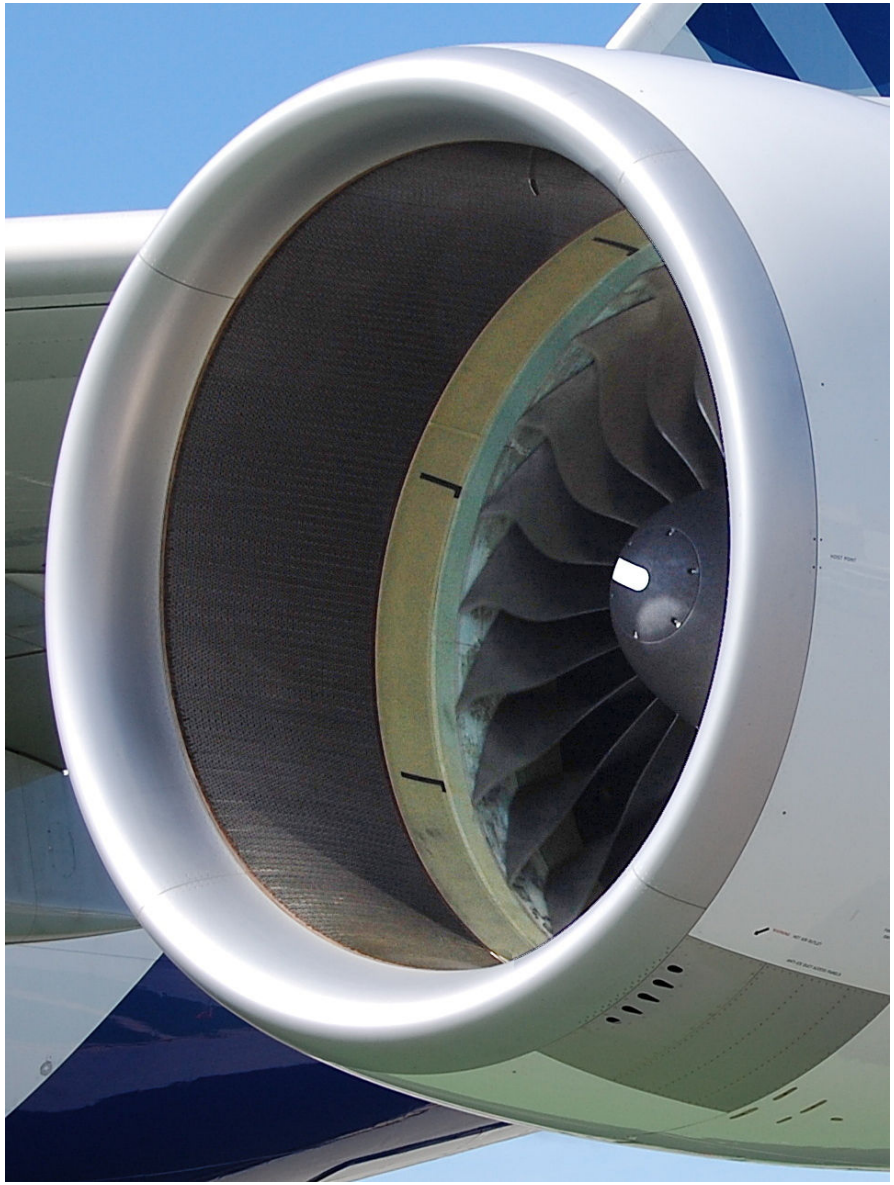
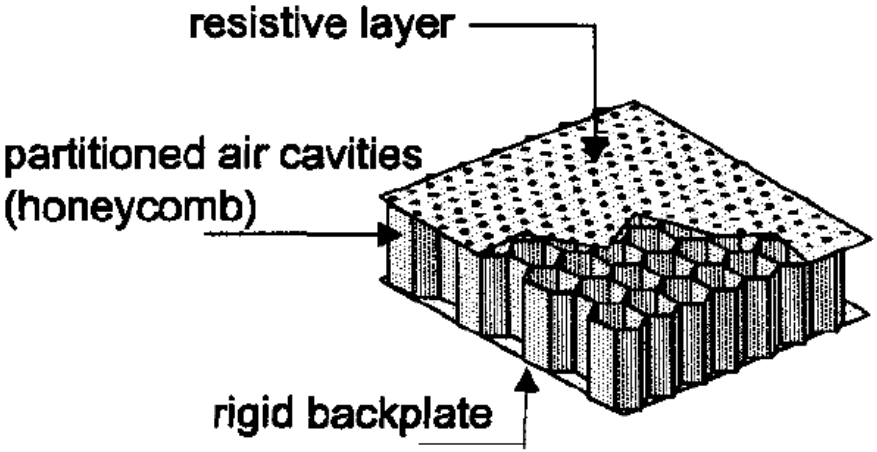


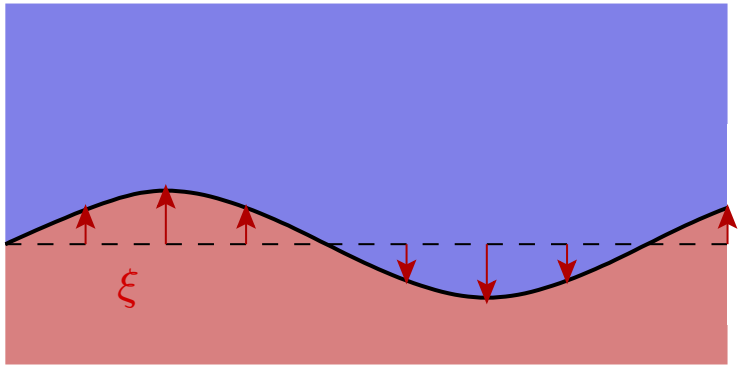
Acoustic linings



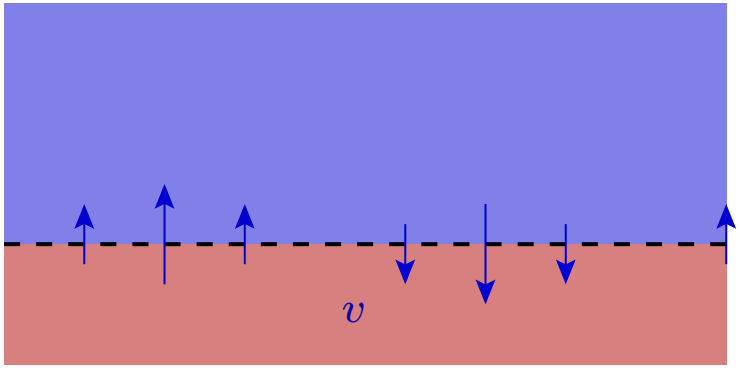


The impedance of a surface

Compliant



Permeable



- Suppose a boundary with velocity $v = \partial\xi/\partial t$ obeys

$$d\frac{\partial^2\xi}{\partial t^2} = -K\xi - R\frac{\partial\xi}{\partial t} + T\frac{\partial^2\xi}{\partial x^2} - B\frac{\partial^4\xi}{\partial x^4} + p.$$

- If $p = \tilde{p}\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ and $v = \tilde{v}\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$,

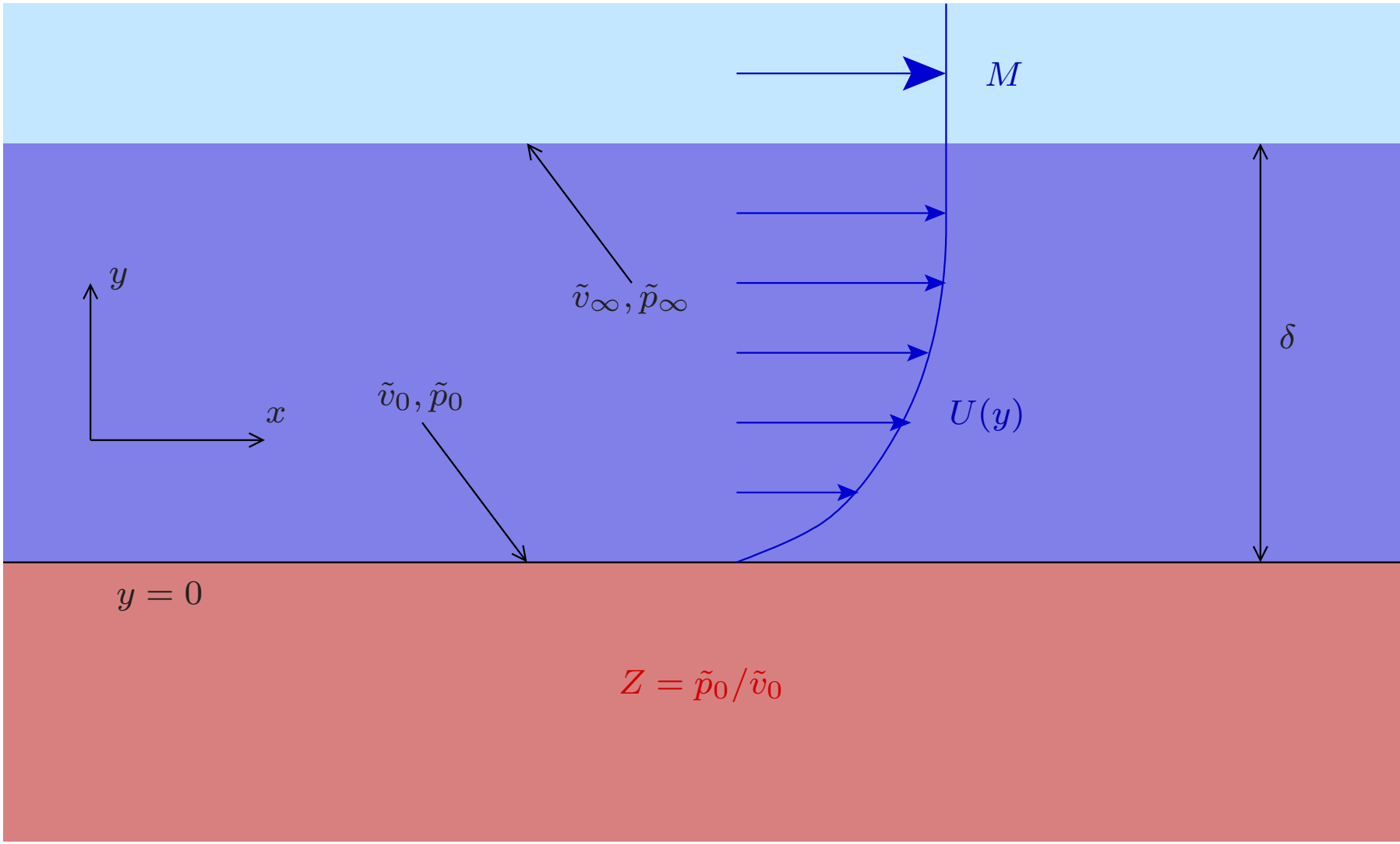
$$\frac{\tilde{p}}{\tilde{v}} = Z = R + \mathrm{i}\left(d\omega - \frac{K}{\omega} - \frac{Tk^2}{\omega} - \frac{Bk^4}{\omega}\right)$$

- Setting bending stiffness B and tension T to zero gives a mass–spring–damper model.
  - No k dependence: “locally reacting”.

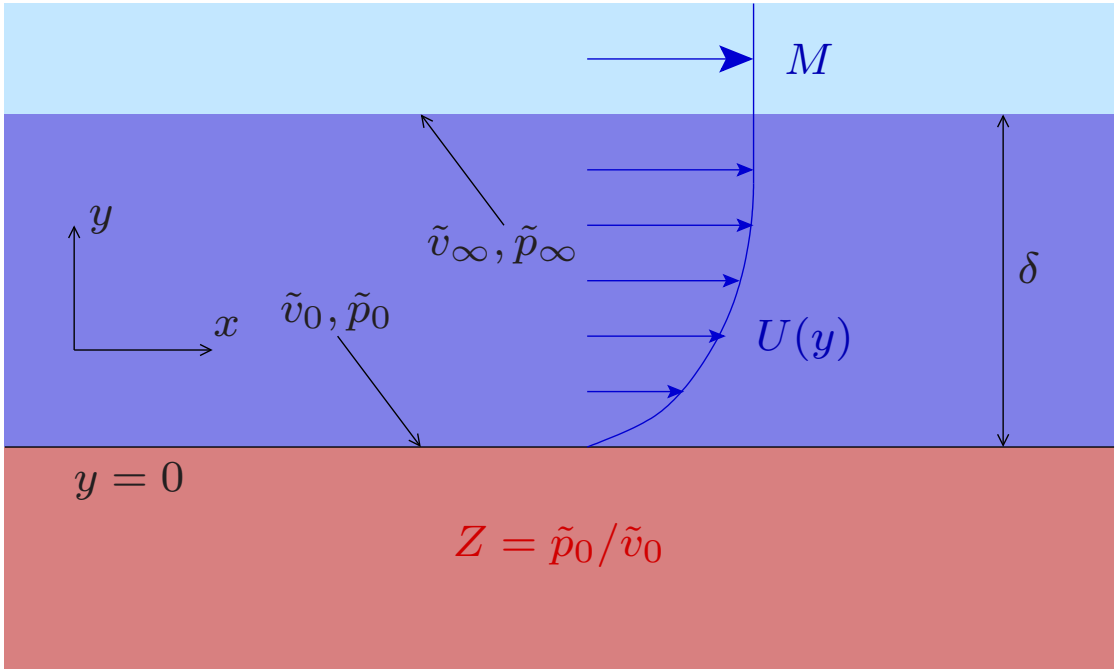
- For the Extended Helmholtz Resonator (EHR) model (Rienstra, 2006 AIAA Paper),

$$Z = R + \mathrm{i}d\omega - \mathrm{i}\nu\cot(\omega L - \mathrm{i}\varepsilon/2).$$

Flow over an impedance surface



Flow over an impedance surface



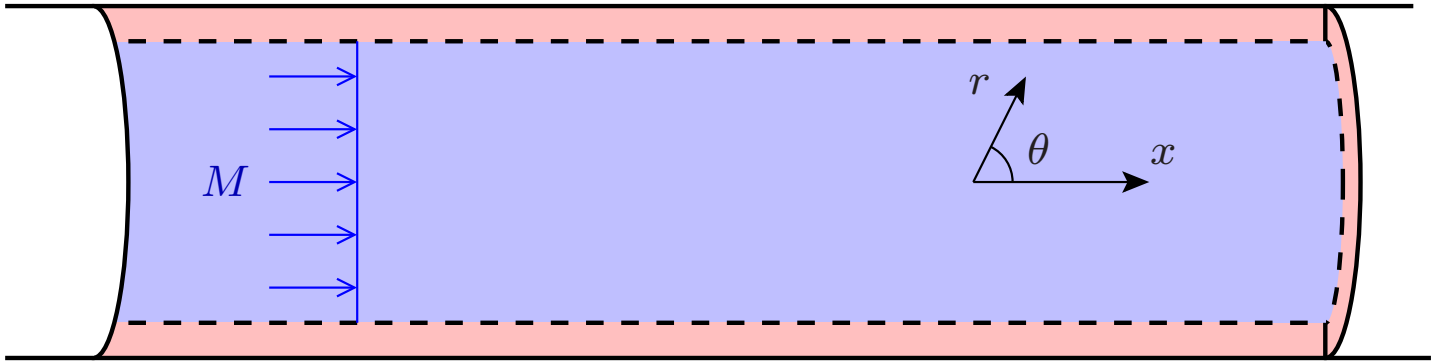
If $\tilde{v} = \mathrm{i}(\omega - Uk)\tilde{\xi}$, then

$$\frac{\mathrm{d}\tilde{p}}{\mathrm{d}y} = O(\delta), \qquad \frac{\mathrm{d}\tilde{\xi}}{\mathrm{d}y} = \frac{Q_T}{\mathrm{i}(\omega - Uk)T} - \frac{\mathrm{i}kQ_u}{(\omega - Uk)^2} + O(\delta).$$

For inviscid flow (Eversman & Beckemeyer 1972 JASA; Tester 1973 JSV; Myers 1980 JSV), or at high frequency (Aurégan *et al.* 2001 JASA, Brambley 2011 JFM), expect

$$\frac{\tilde{p}_\infty}{\tilde{v}_\infty} = \frac{\tilde{p}_\infty}{\mathrm{i}(\omega - Mk)\tilde{\xi}} = \frac{\omega Z}{\omega - Mk}$$

Sound in a cylinder with uniform flow



Linearized Euler equations (nondimensionalized so $c^2 = 1$):

$$\boldsymbol{u} = M\boldsymbol{e_x} + \boldsymbol{\nabla}\phi \qquad\qquad p - p_0 = \rho - \rho_0 = -\frac{\mathrm{D}\phi}{\mathrm{D}t}$$

$$\frac{\mathrm{D}^2\phi}{\mathrm{D}t^2} = \nabla^2\phi.$$

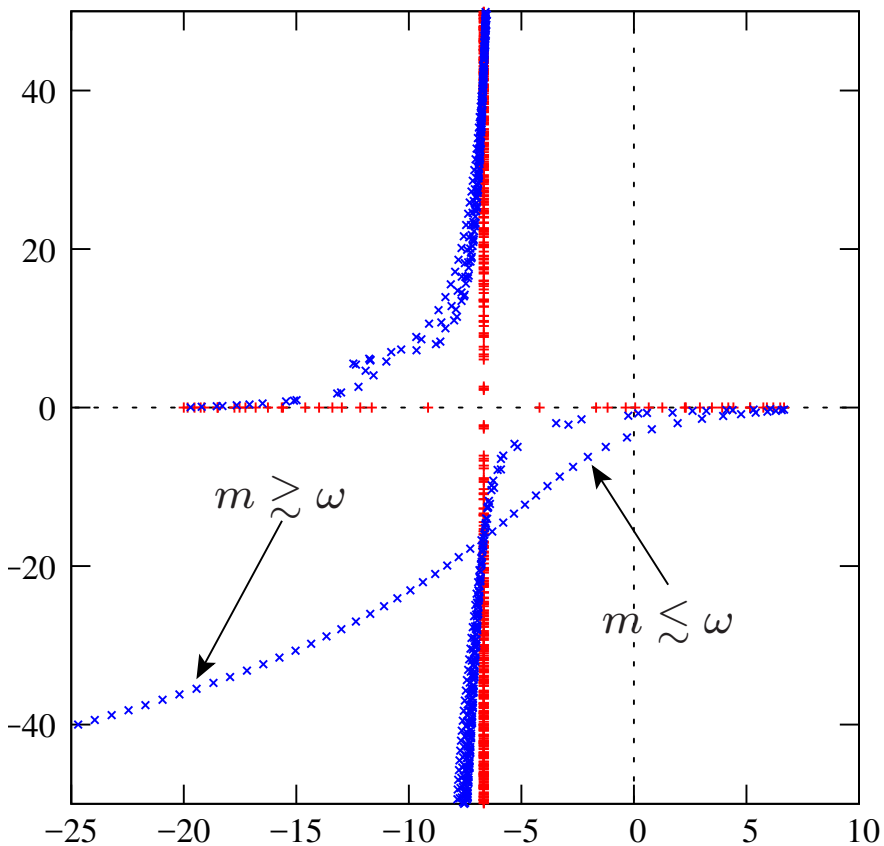
Solution:

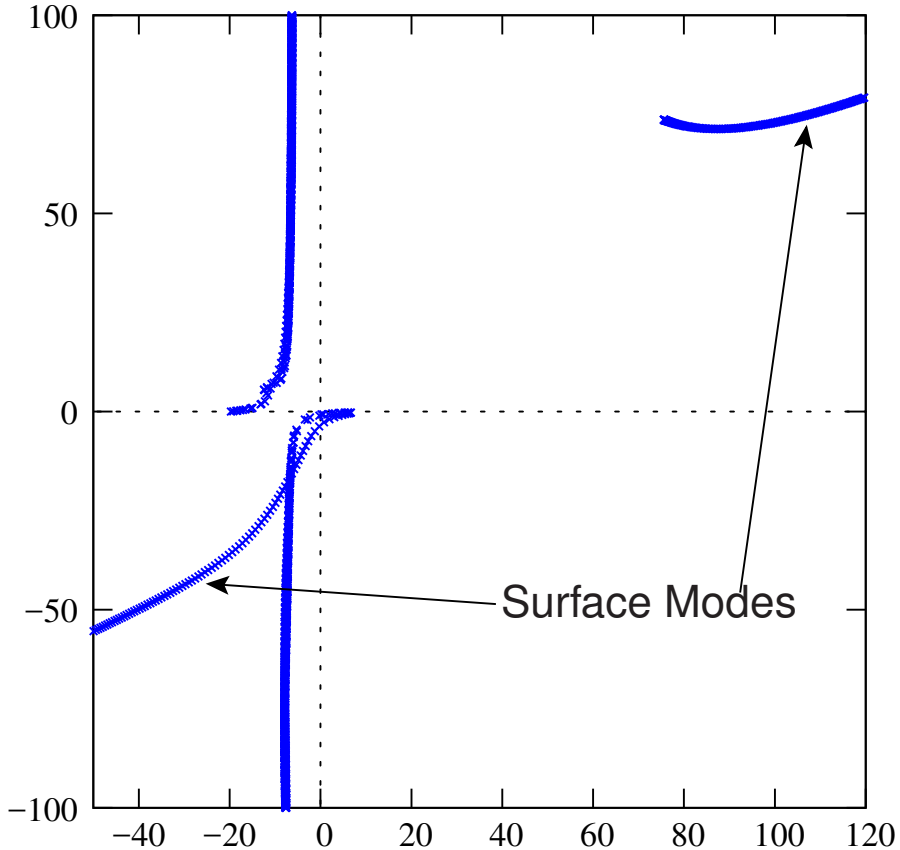
$$\phi = A\mathrm{J}_m(\alpha r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\} \qquad p = -\mathrm{i}(\omega - Mk)\phi \qquad \alpha^2 = (\omega - Mk)^2 - k^2$$

Myers impedance boundary condition:

$$D(k,\omega) \equiv \mathrm{i}\omega Z\alpha\mathrm{J}_m'(\alpha) - (\omega - Mk)^2\mathrm{J}_m(\alpha) = 0.$$

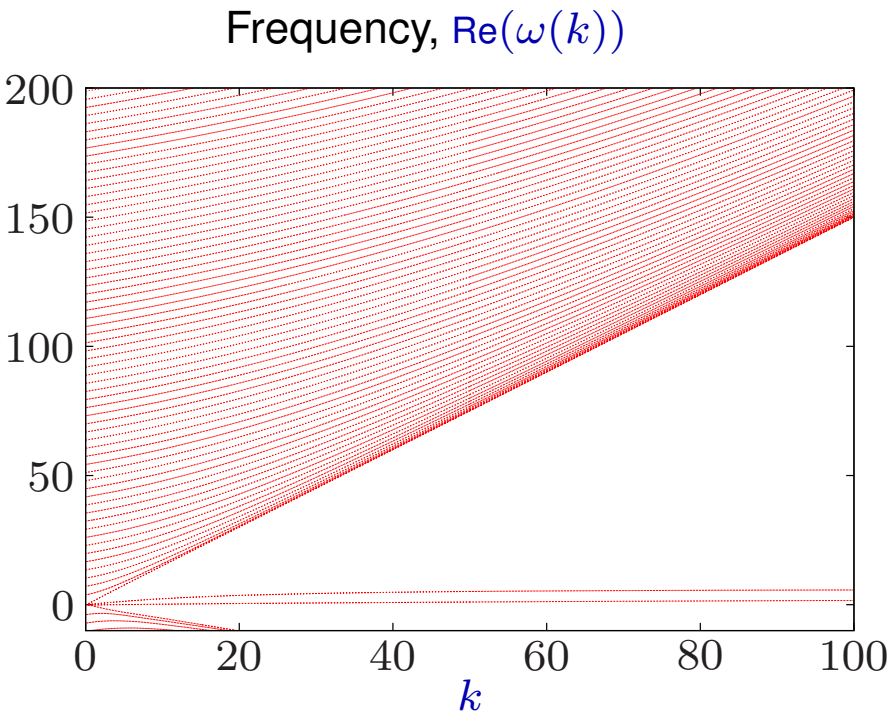
Boundary value problem (given ω, find k)

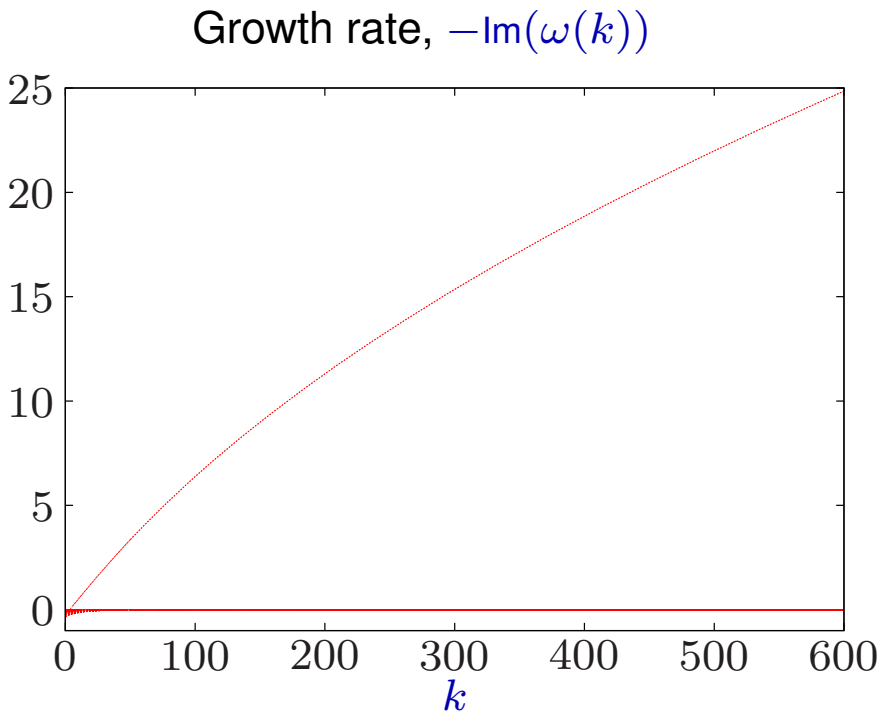




- $M = 0.5$, $\omega = 10$, $Z = 2 - \mathrm{i}$.
- Surface modes: Rienstra (2003, WM), Brambley & Peake (2006, WM).

Initial value problem (given k, find ω)





- Unbounded exponential growth rate $-\mathrm{Im}(\omega(k)) \sim k^{1/2}$.
- Mass–spring–damping boundary $Z = \mathrm{i}d\omega - \mathrm{i}b/\omega + R$.
- This behaviour can be predicted in general (Brambley, 2009 JSV).

Modified Boundary condition (1st order correction in δ)

- For uniform flow,

$$\phi = A\mathrm{J}_m(\alpha r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\} \qquad \text{where} \qquad \alpha^2 = (\omega - Mk)^2 - k^2.$$

- Myers boundary condition (asymptotically correct to $o(\delta)$):

$$\mathrm{i}\omega Z\alpha\mathrm{J}_m'(\alpha) - \big(\omega - Mk\big)^2\mathrm{J}_m(\alpha) = 0.$$

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- New modified boundary condition (asymptotically correct to $o(\delta^2)$):

$$\mathrm{i}\big(\omega \qquad\qquad\big)Z\big[\alpha\mathrm{J}_m'(\alpha) \qquad\qquad\qquad\qquad\qquad\qquad\big] - (\omega - Mk)^2\big[\mathrm{J}_m(\alpha) \qquad\qquad\qquad\big] = 0.$$

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- New modified boundary condition (asymptotically correct to $o(\delta^2)$):

$$\mathrm{i}\big(\omega - U(1)k\big)Z\big[\alpha\mathrm{J}'_m(\alpha) \qquad \big] - (\omega - Mk)^2\big[\mathrm{J}_m(\alpha) \qquad \big] = 0.$$

Modified Boundary condition (1st order correction in δ)

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$$\phi = A\mathrm{J}_m(\alpha r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\} \qquad \text{where} \qquad \alpha^2 = (\omega - Mk)^2 - k^2.$$

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$$\mathrm{i}\big(\omega - U(1)k\big)Z\big[\alpha\mathrm{J}_m'(\alpha) \qquad \big] - (\omega - Mk)^2\big[\mathrm{J}_m(\alpha) - \alpha\mathrm{J}_m'(\alpha)\delta I_0\big] = 0.$$

where

$$\delta I_0 = \int_0^1 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r,$$

Modified Boundary condition (1st order correction in δ)

- For uniform flow,

$$\phi = A\mathrm{J}_m(\alpha r)\exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\} \qquad \text{where} \qquad \alpha^2 = (\omega - Mk)^2 - k^2.$$

- Myers boundary condition (asymptotically correct to $o(\delta)$):

$$\mathrm{i}\omega Z\alpha\mathrm{J}_m'(\alpha) - \big(\omega - Mk\big)^2\mathrm{J}_m(\alpha) = 0.$$

- New modified boundary condition (asymptotically correct to $o(\delta^2)$):

$$\mathrm{i}\big(\omega - U(1)k\big)Z\big[\alpha\mathrm{J}_m'(\alpha) - \big(k^2 + m^2\big)\delta I_1\mathrm{J}_m(\alpha)\big] - (\omega - Mk)^2\big[\mathrm{J}_m(\alpha) - \alpha\mathrm{J}_m'(\alpha)\delta I_0\big] = 0.$$

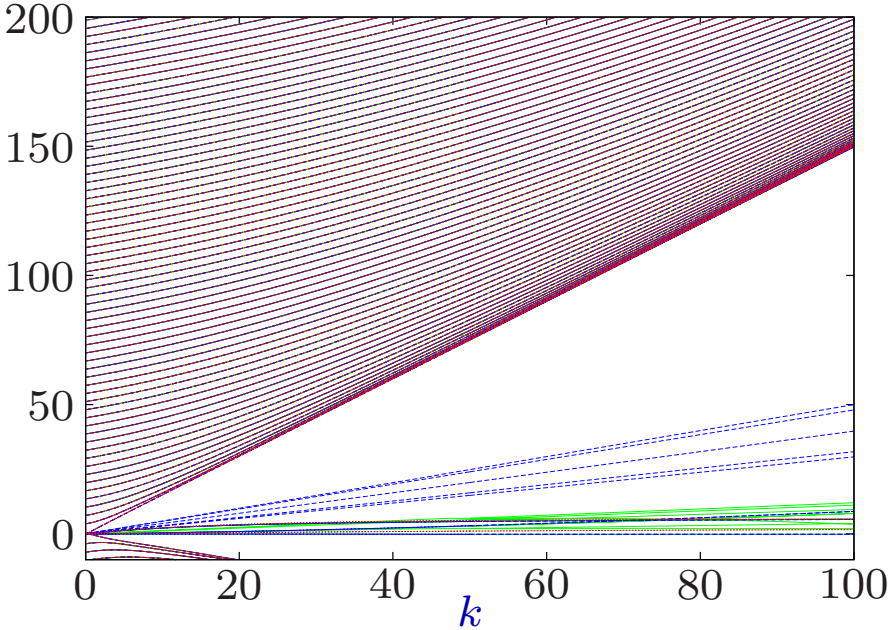
where

$$\delta I_0 = \int_0^1 1 - \frac{\big(\omega - U(r)k\big)^2 R(r)}{(\omega - Mk)^2}\,\mathrm{d}r, \qquad \delta I_1 = \int_0^1 1 - \frac{(\omega - Mk)^2}{\big(\omega - U(r)k\big)^2 R(r)}\,\mathrm{d}r.$$

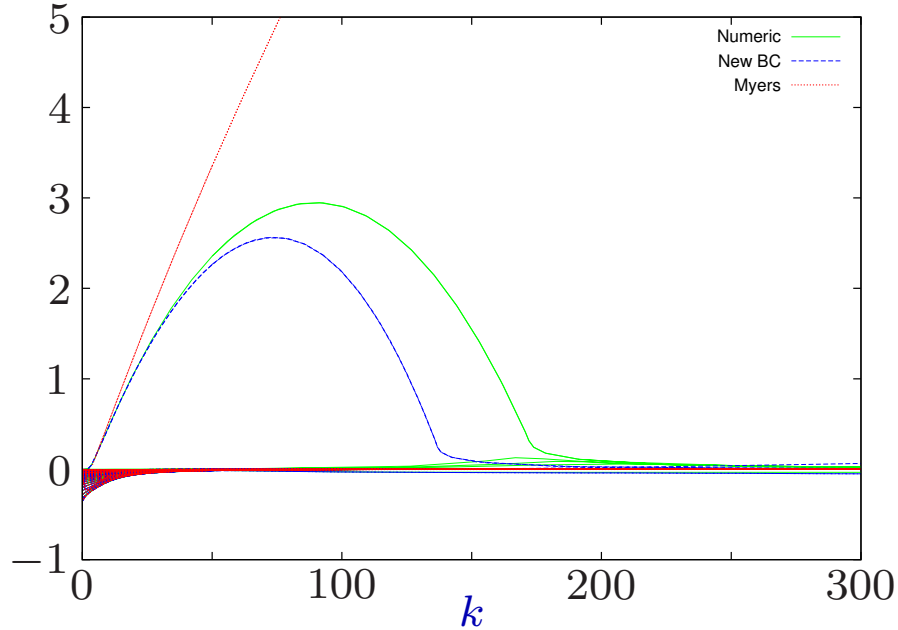
- See Brambley (2011 AIAA J) for details.

Stability of the modified boundary condition

Frequency, $\mathrm{Re}(\omega(k))$

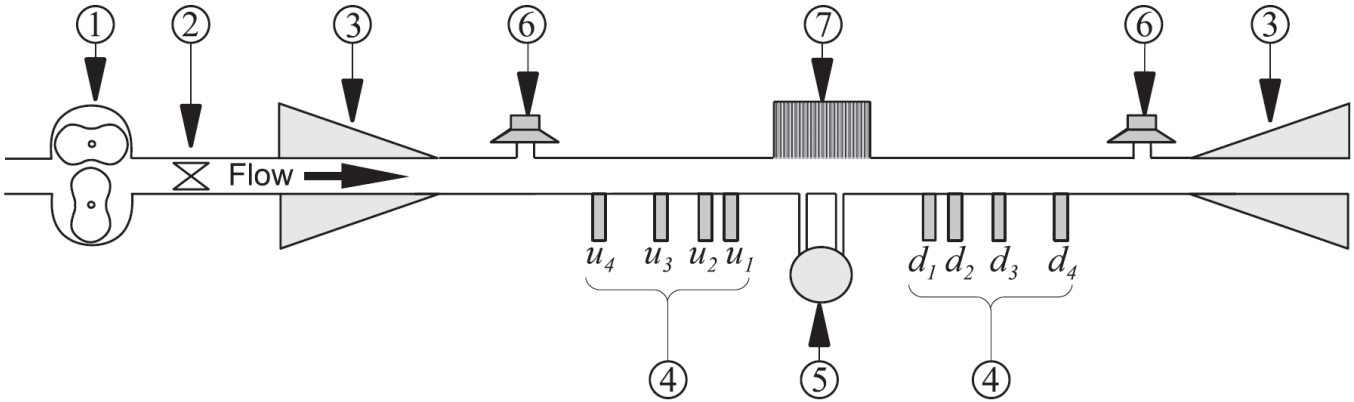


Growth rate, $-\mathrm{Im}(\omega(k))$

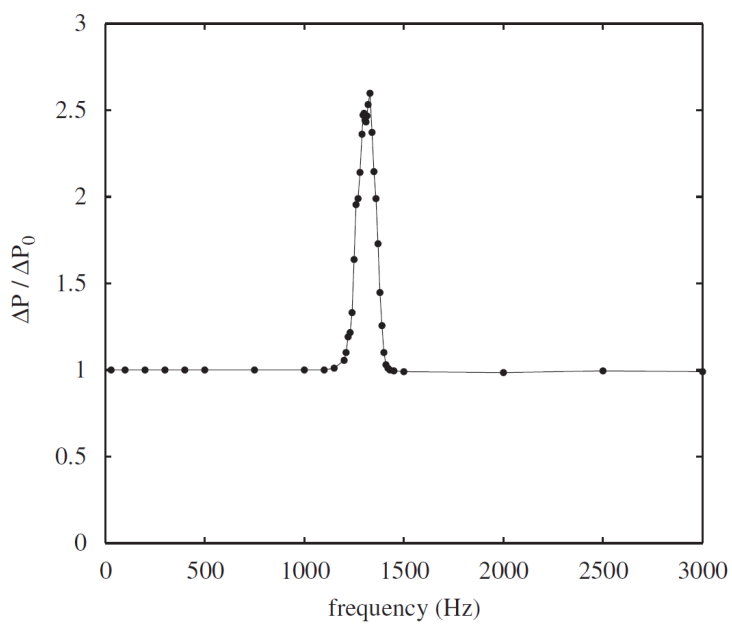


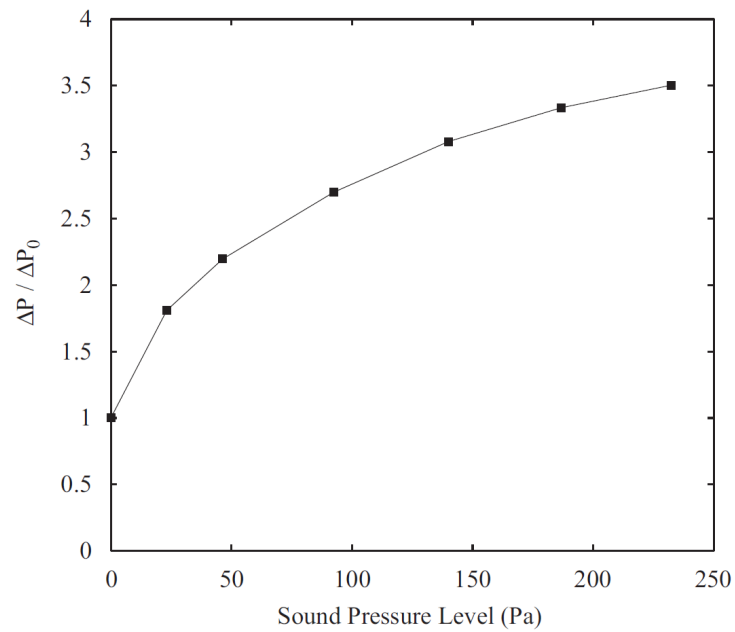
- $m = 0, \quad Z = 3 + 0.15\mathrm{i}\omega - 1.15\mathrm{i}/\omega.$
- $U(r) \approx M \tanh\big((1-r)/\delta\big)$ with $M = 0.5$ and $\delta = 2 \times 10^{-4}$.

Experimental evidence of instability



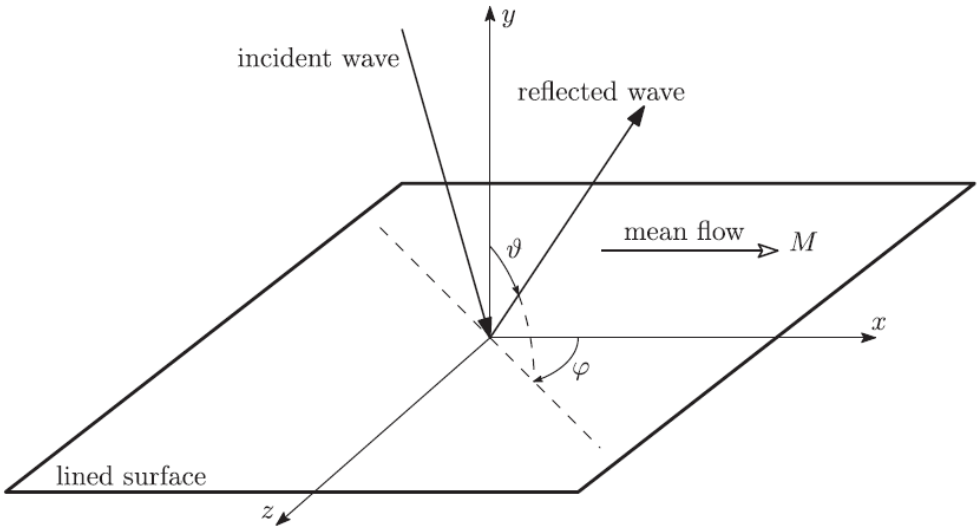
1: compressor, 2: flowmeter, 3: anechoic terminations, 4: microphones, 5: static pressure measurement, 6: acoustical source, 7: lined wall.

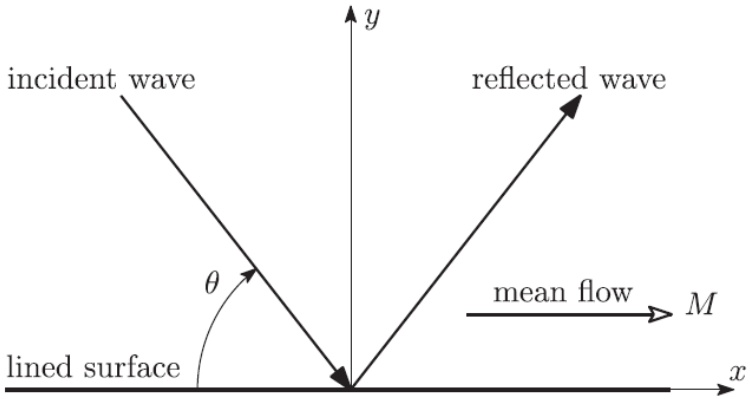




Taken from Aurégan & Leroux (2008, JSV).

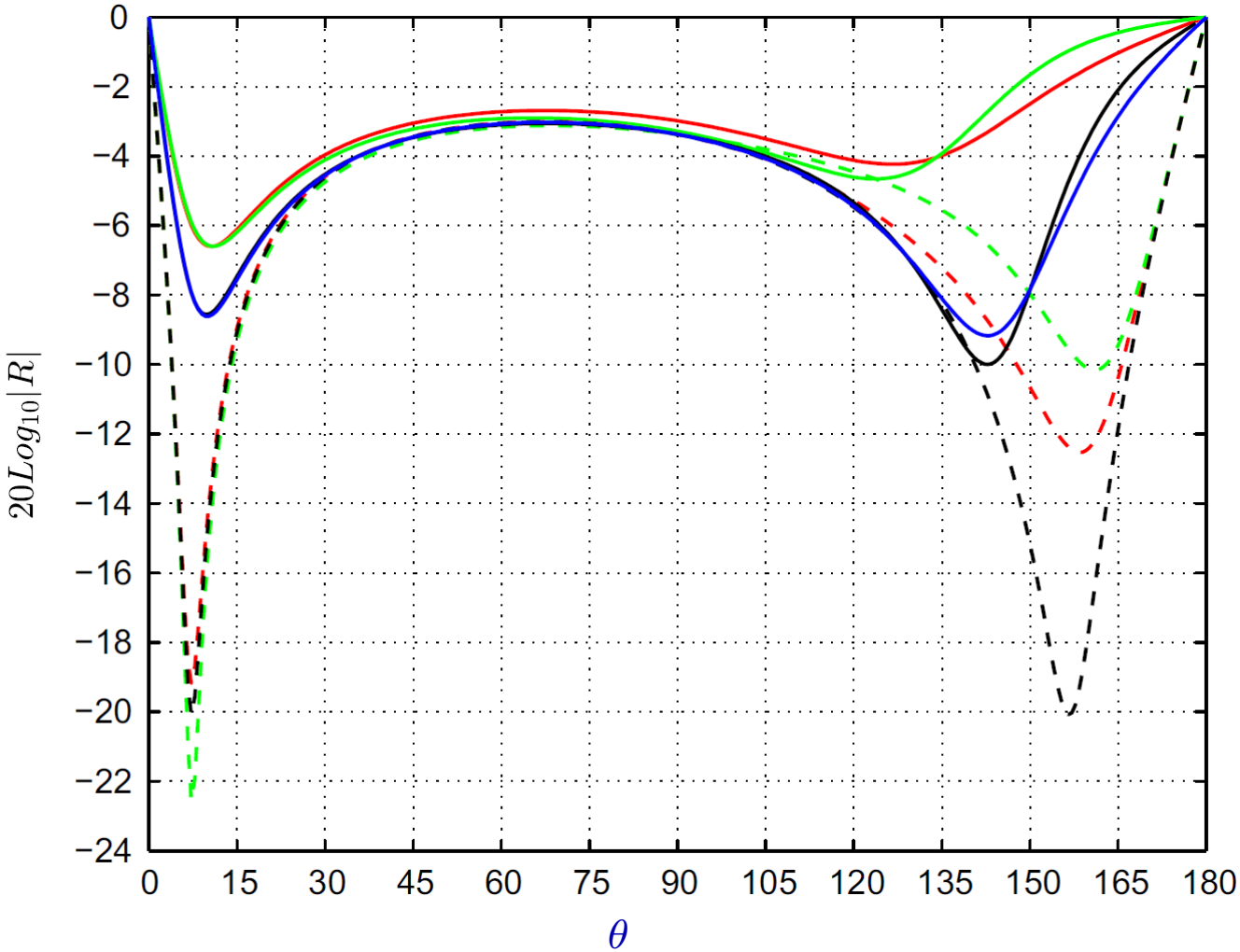
Accuracy (from Gabard, 2013 JSV)





- $\omega = 28$, $Z = 5 - \mathrm{i}$, $M = 0.55$, $\delta = 0.014 \Rightarrow \omega\delta = 0.39$.

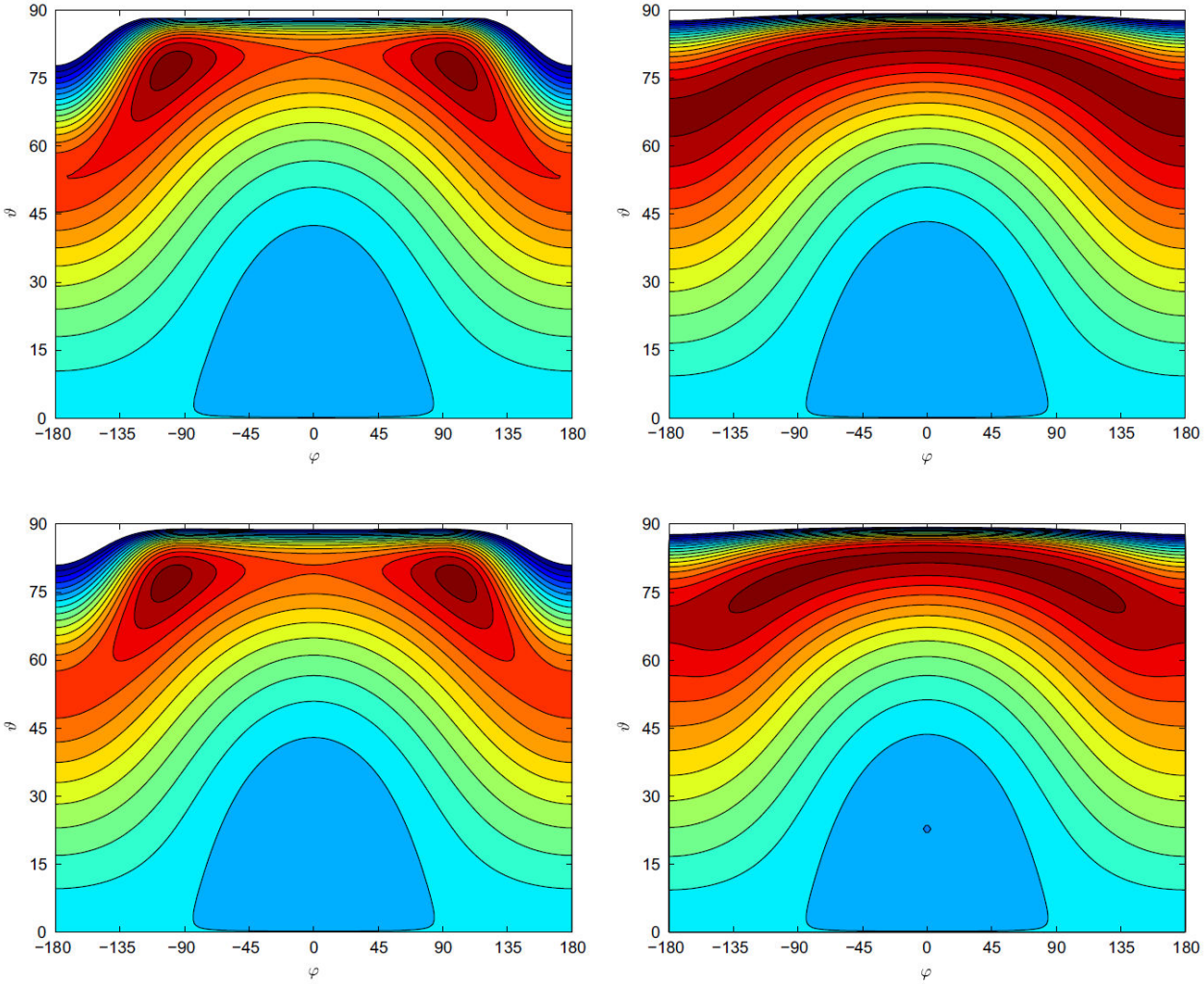
Accuracy (from Gabard, 2013 JSV)



- Solid: Exact. Dashed: Myers. Blue: Brambley (2011 AIAA J).
  Green & Red: [redacted] (2011, JFM).

Accuracy (from Gabard, 2013 JSV)

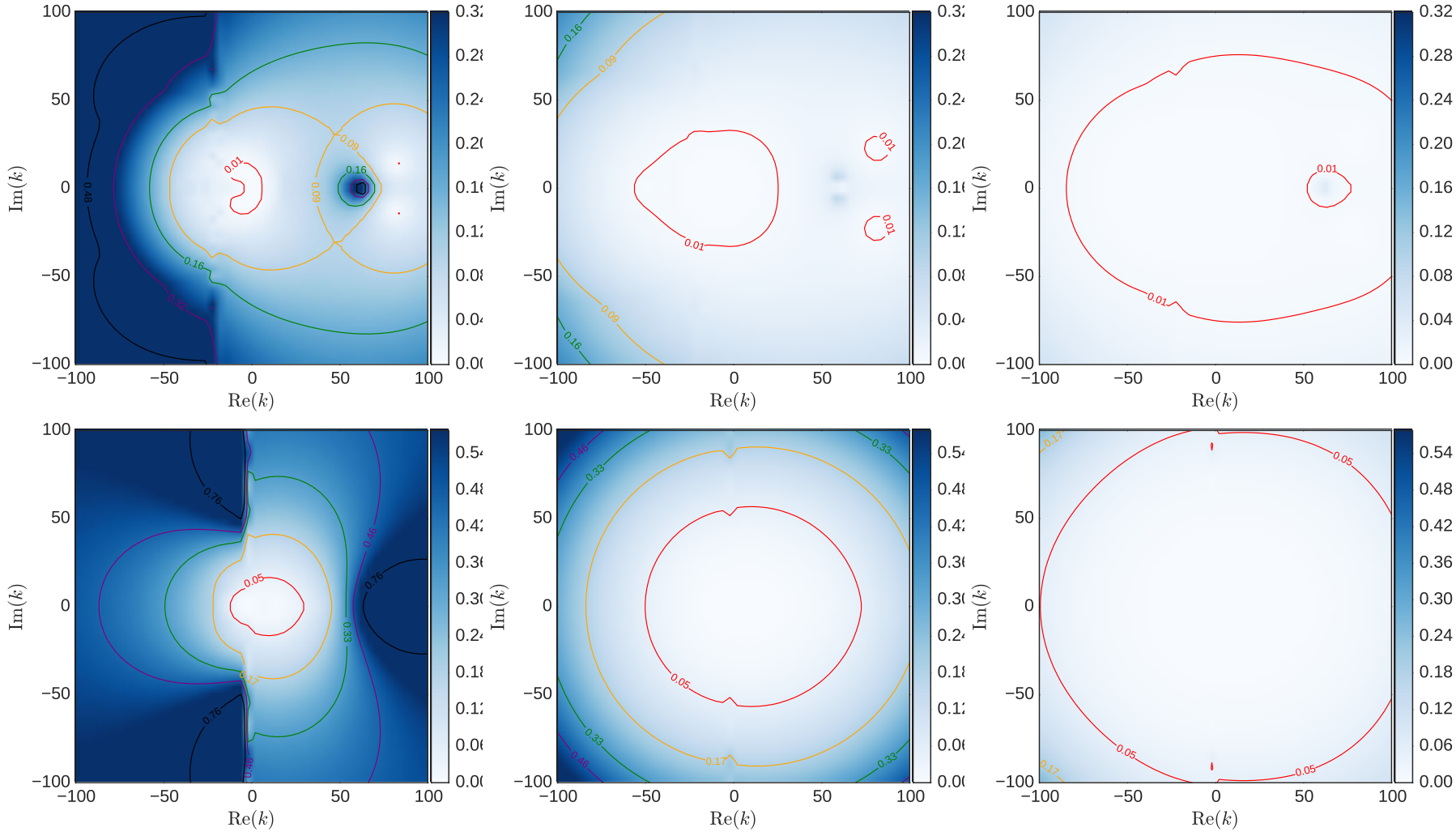
Exact | Myers



Brambley (2011 AIAA J)

[redacted] (2011, JFM)

Accuracy (Khamis & Brambley, JFM 2016)



Myers BC | Modified BC | 2nd order BC

- Error is given by $\min\left(|Z_1 - Z_2|, |1/Z_1 - 1/Z_2|\right)$.

Viscosity

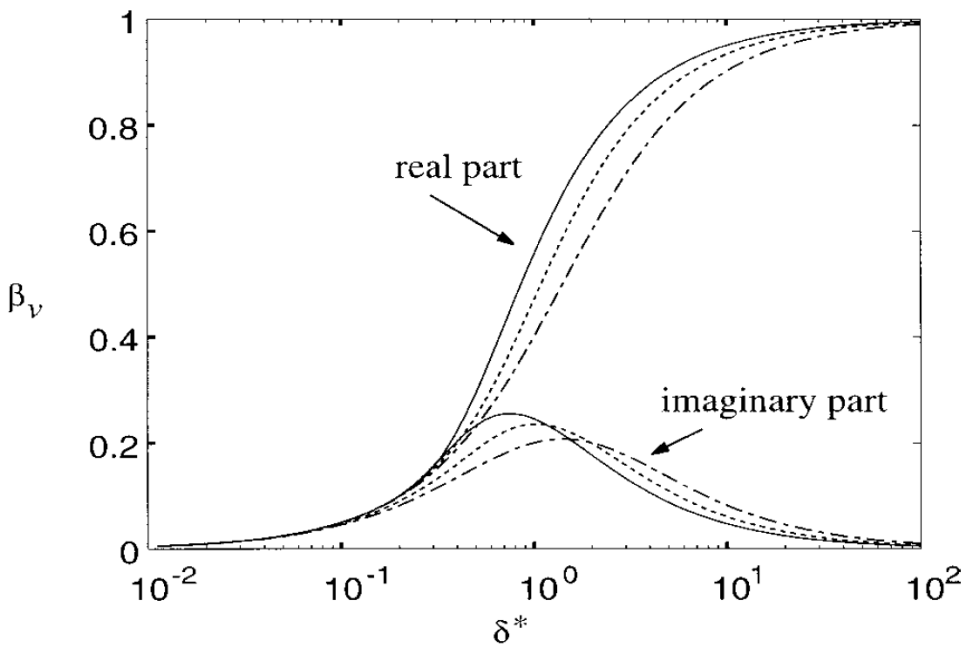
Aurégan, Starobinski & Pagneux (2001, JASA):

$$\frac{p_\infty}{v_\infty} = \left(1 - (1-\beta_v)\frac{Mk}{\omega} + \left(\frac{T_0 - T(0)}{T(0)}\right)\beta_t\right)^{-1} Z$$

$$\beta_v = \frac{1}{M}\int_0^\infty U_y(y)\exp\left\{-y\sqrt{\mathrm{i}\rho(0)\omega}\right\}\mathrm{d}y$$

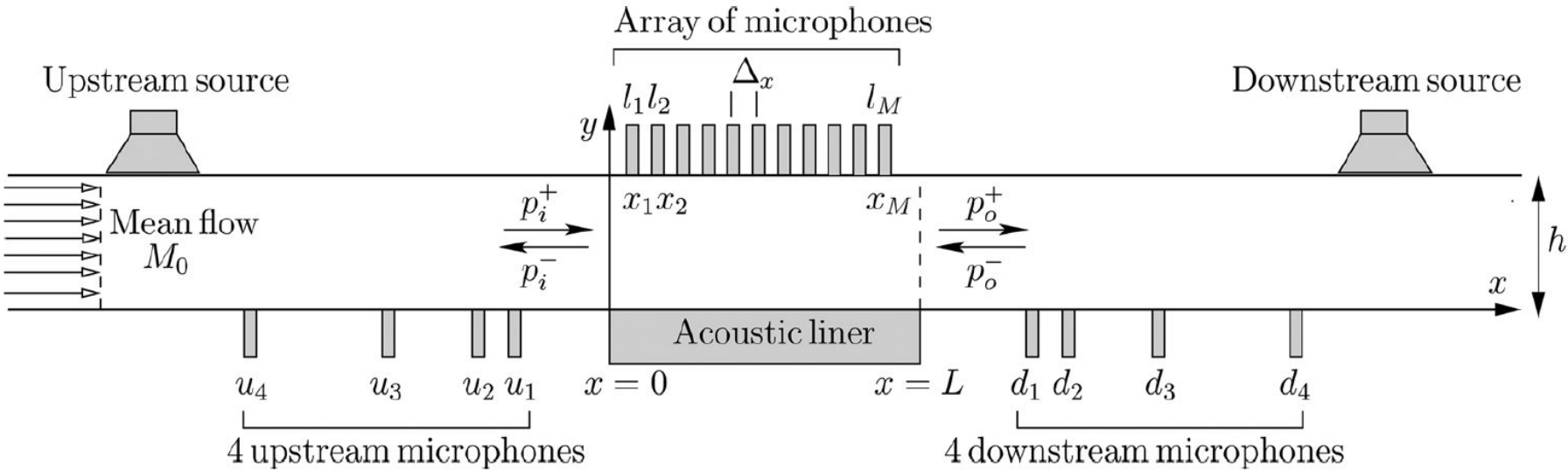
$$\beta_t = \frac{1}{T_0 - T(0)}\int_0^\infty T_y(y)\exp\left\{-y\sqrt{\mathrm{i}\rho(0)\mathrm{Pr}\omega}\right\}\mathrm{d}y$$

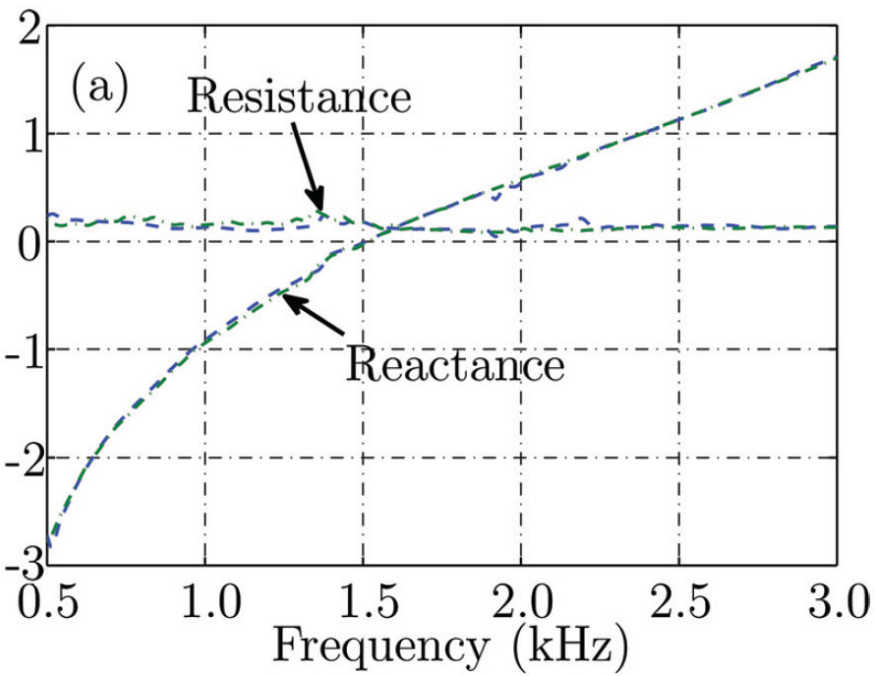
High frequency $\Rightarrow$ $\beta_v \to 0$ and $\beta_t \to 0$ gives continuity of normal displacement.
Low frequency $\Rightarrow$ $\beta_v \to 1$ and $\beta_t \to 1$ gives continuity of normal mass flux.

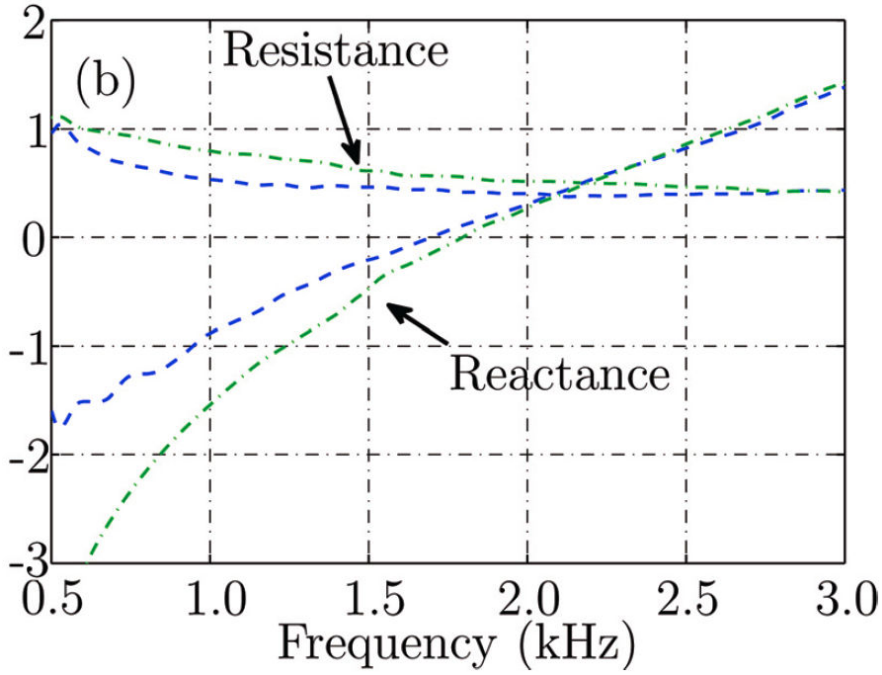


$$\delta^* = \frac{U_y(0)}{M\sqrt{\omega\rho(0)/2}}$$

Importance of Viscosity (Renou & Aurégan, 2011 JASA)







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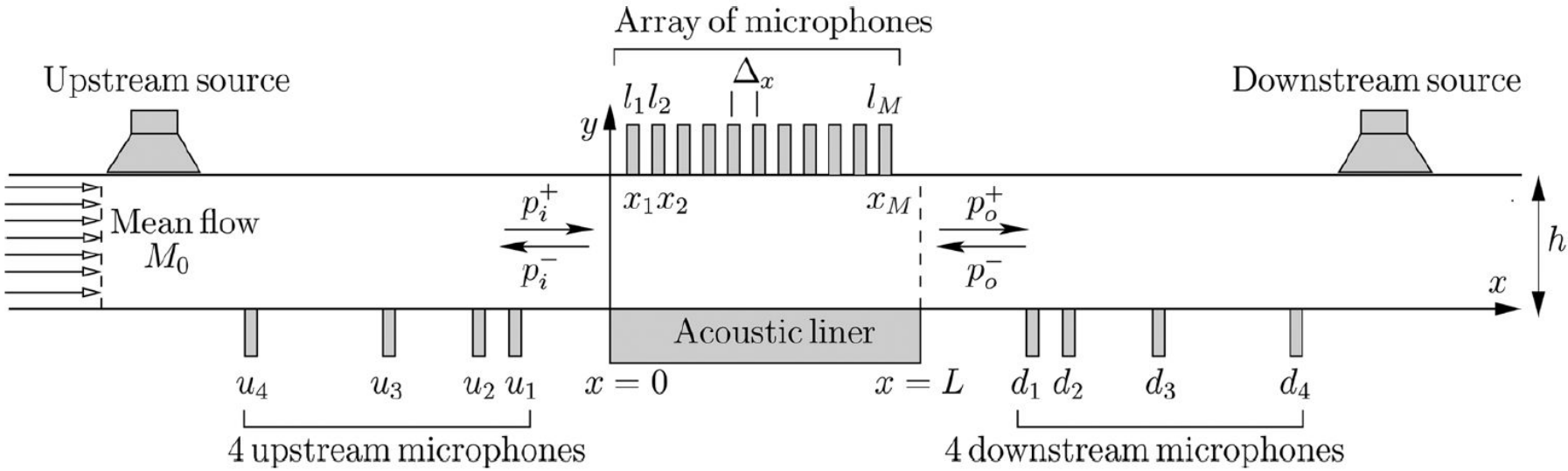
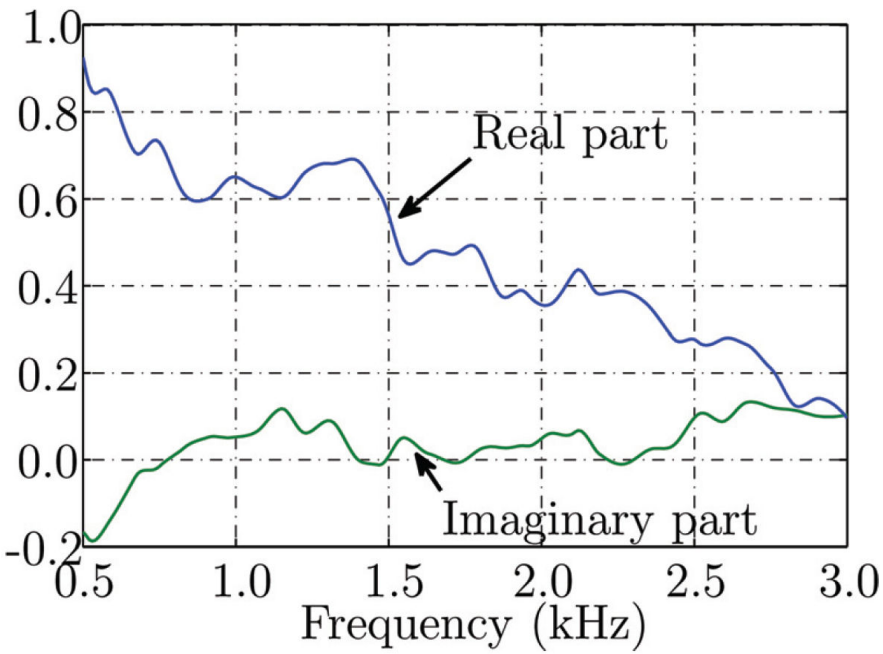
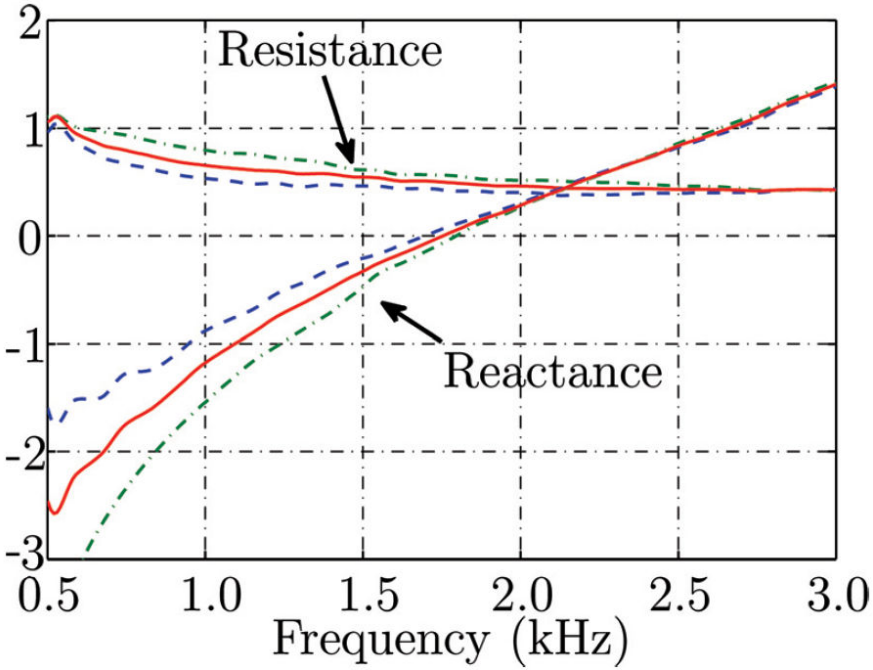






Viscous compressible governing equations

- Nondimensionalize based on a lengthscale (e.g. duct radius), the speed of sound, and the centreline density.
  - At $r = 0$, find $1/\mu = M\mathrm{Re}$ and $1/\kappa = M\mathrm{RePr}$.

- Dimensionless governing equations:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0$$

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \sigma$$

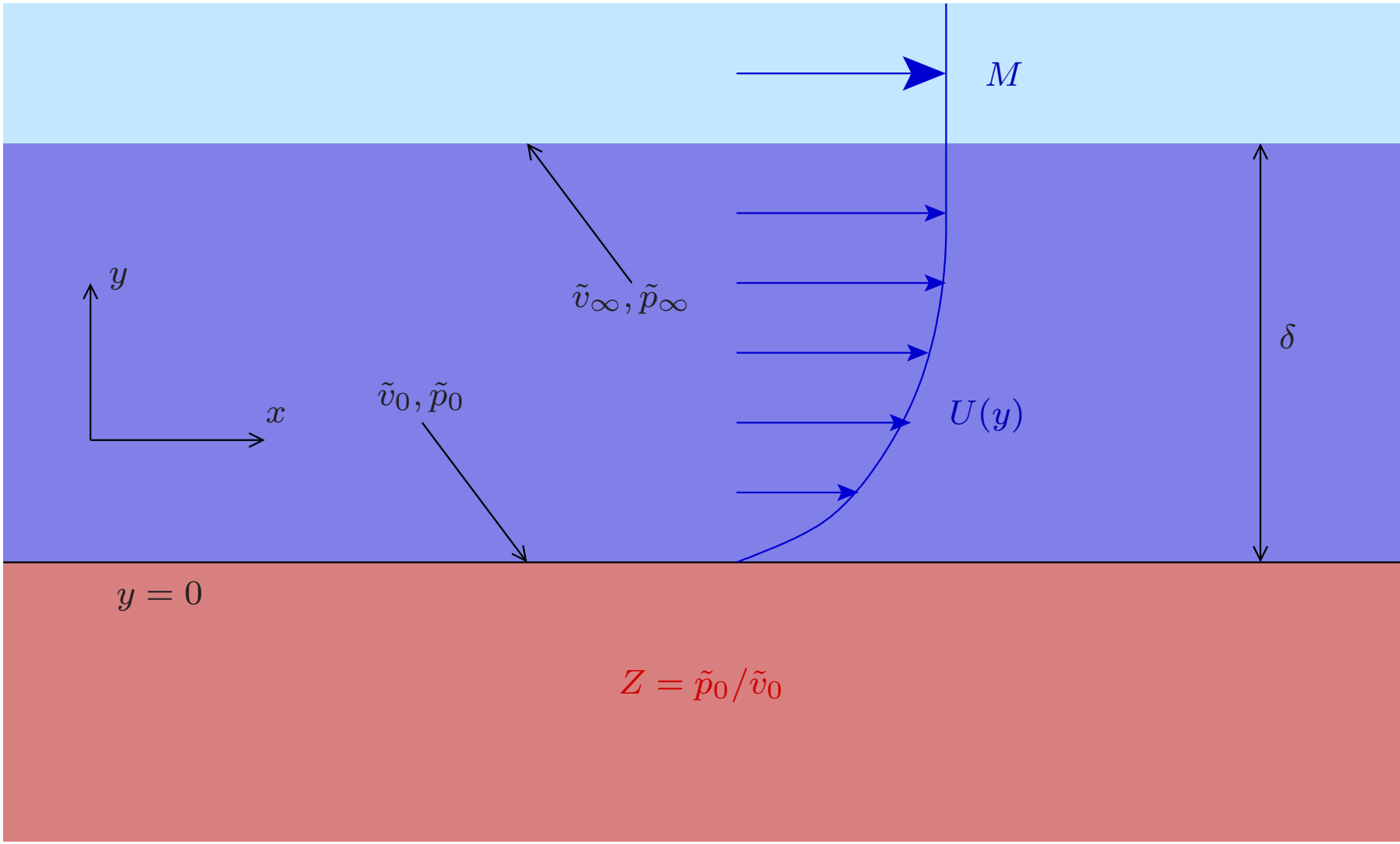
$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \left(\mu^{\mathrm{B}} - \frac{2}{3}\mu \right) \delta_{ij} \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$$\rho \frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + \sigma_{ij} \frac{\partial u_i}{\partial x_j} + \boldsymbol{\nabla} \cdot (\kappa \boldsymbol{\nabla} T)$$

$$T = \frac{\gamma}{\gamma - 1} \frac{p}{\rho}$$

$\mu, \mu^{\mathrm{B}}, \kappa$ linear in T and independent of p.

Viscosity within the boundary layer



Viscosity within the boundary layer

- Linearize about a steady flow and rescale inside boundary layer ($\delta = \mu(0)^{-1/2}$):

$$r = 1 - \delta y \qquad \boldsymbol{u} = (U + \varepsilon\tilde{u})\boldsymbol{e_x} - \delta\varepsilon\tilde{v}\boldsymbol{e_r} + \varepsilon\tilde{w}\boldsymbol{e_\theta} \qquad \varepsilon = |\varepsilon| \exp\{\mathrm{i}\omega t - \mathrm{i}kx - \mathrm{i}m\theta\}$$

- To leading order in δ, linearized Navier–Stokes gives

$$\tilde{p} = \text{constant} \qquad \tilde{w} = \text{unconnected} \qquad \tilde{\rho} = -(\gamma - 1)\rho^2\tilde{T}$$

$$\mathrm{i}(\omega - Uk)\tilde{T} + T_y\tilde{v} - T\tilde{v}_y + \mathrm{i}kT\tilde{u} = 0,$$

$$\mathrm{i}(\omega - Uk)\tilde{u} + U_y\tilde{v} = (\gamma - 1)^2 T\left(T\tilde{u}_y + \tilde{T}U_y\right)_y,$$

$$\mathrm{i}(\omega - Uk)\tilde{T} + T_y\tilde{v} = (\gamma - 1)^2 T\left[\frac{1}{\mathrm{Pr}}\left(\tilde{T}T\right)_{yy} + \tilde{T}U_y^2 + 2TU_y\tilde{u}_y\right].$$

- Boundary conditions for a fixed permeable boundary to leading order in δ:

$$\tilde{u}(0) = 0, \qquad \tilde{v}(0) = -\tilde{v}_0/\delta, \qquad \tilde{T}(0) = 0,$$

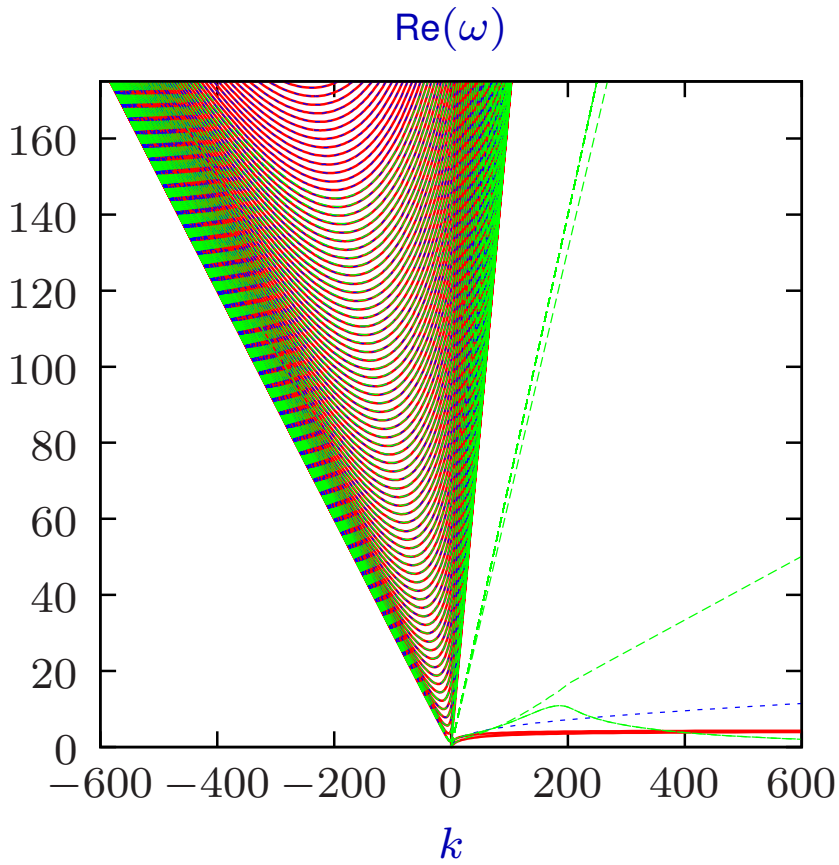
$$\tilde{u}(y) \to 0, \qquad \tilde{v}(y) \to -\tilde{v}_\infty/\delta, \qquad \tilde{T}(y) \to 0, \qquad \text{as } y \to \infty.$$

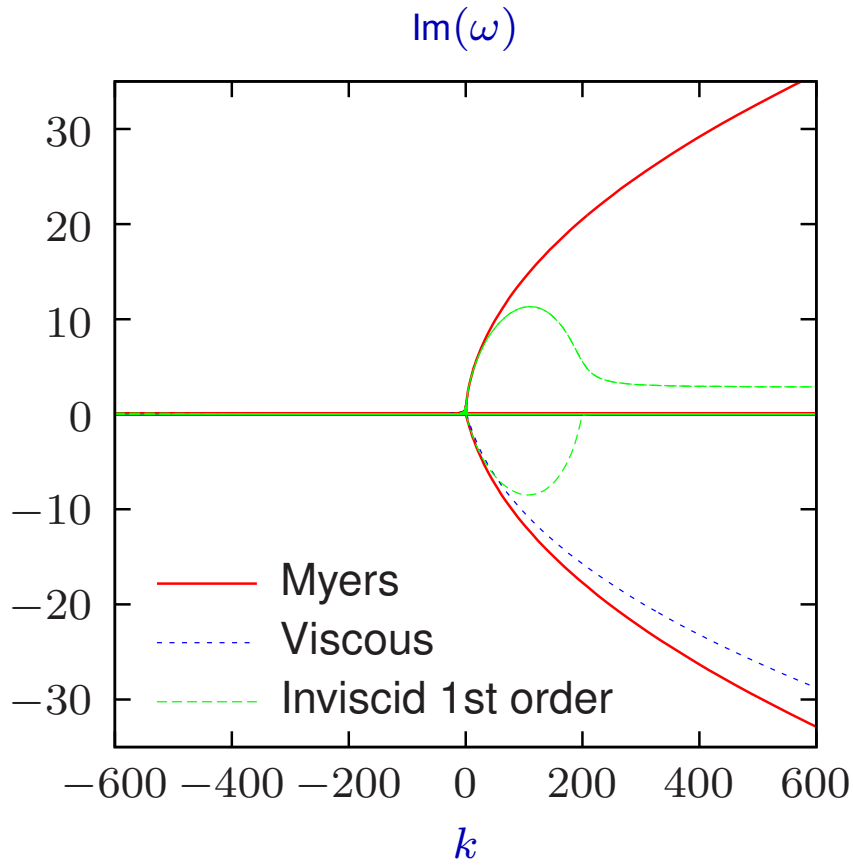
- Note: For an impermeable flexible boundary, get

$$\tilde{u}(0) = \frac{u_y(0)\tilde{v}}{\mathrm{i}\omega}.$$

- For details, see Brambley (2011 JFM).

But...





- For Myers, $-\mathrm{Im}(\omega) = O(k^{1/2})$. For viscous leading order, $-\mathrm{Im}(\omega) = O(k^{1/3})$.
- See Brambley (2011, JFM) for details.

Next order Viscous (Khamis & Brambley, 2017 JFM)

Linearized nondimensionalized governing equations:

$$\mathrm{i}(\omega - Uk)\hat{T} + ikT\hat{u} - T^2\left(\frac{\hat{v}}{T}\right)_y = \delta\left[\gamma\mathrm{i}(\omega - Uk)T\tilde{p} - T\hat{v} - imT\tilde{w}\right],$$

$$\mathrm{i}(\omega - Uk)\hat{u} + U_y\hat{v} - (\gamma-1)^2T(T\hat{u}_y + U_y\hat{T})_y = \delta\left[\mathrm{i}(\gamma-1)kT\tilde{p} - (\gamma-1)^2T(T\hat{u}_y + U_y\hat{T})\right],$$

$$\tilde{p}_y = \delta\left[2(\gamma-1)(T\hat{v}_y)_y + (\gamma-1)\left(\frac{\mu_B^*}{\mu^*} - \frac{2}{3}\right)(T\hat{v}_y - ikT\hat{u})_y - \frac{\mathrm{i}(\omega - Uk)}{(\gamma-1)T}\hat{v} - \mathrm{i}(\gamma-1)k(T\hat{u}_y + U_y\hat{T})\right],$$

$$(T\tilde{w}_y)_y - \frac{\mathrm{i}(\omega - Uk)}{(\gamma-1)^2T}\tilde{w} = \frac{im}{\gamma-1}\tilde{p} + \mathcal{O}(\delta),$$

$$\mathrm{i}(\omega - Uk)\hat{T} + T_y\hat{v} - \frac{(\gamma-1)^2T}{\mathrm{Pr}}(T\hat{T})_{yy} - (\gamma-1)^2T(U_y^2\hat{T} + 2TU_y\hat{u}_y) = \delta\left[(\gamma-1)\mathrm{i}(\omega - Uk)T\tilde{p} - \frac{(\gamma-1)^2T}{\mathrm{Pr}}(T\tilde{T})_y\right],$$

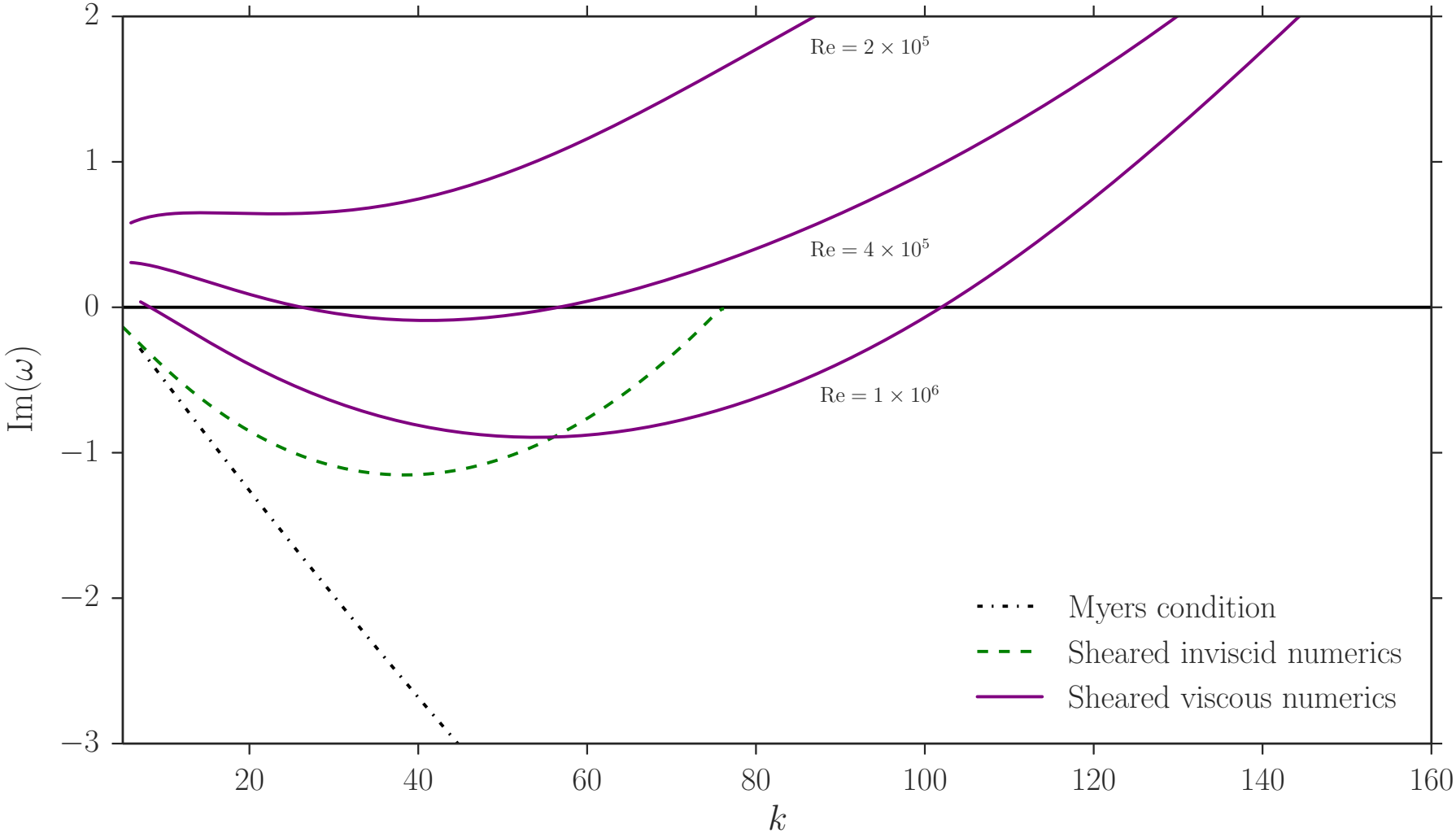
Boundary conditions at $\mathcal{O}(1)$: $\hat{u}_0(0) = 0,\quad \hat{T}_0(0) = 0,\quad -\hat{v}_0(0)Z = \tilde{p}_0,\quad \tilde{w}_0(0) = 0,$

as $y \to \infty$ $\quad \hat{u}_0(y) \to 0,\quad \hat{T}_0(y) \to 0,\quad \tilde{w}_0 \to w_{\mathrm{u}}.$

Boundary conditions at $\mathcal{O}(\delta)$: $\hat{u}_1(0) = 0,\quad \hat{T}_1(0) = 0,\quad \hat{v}_1(0) = 0,$

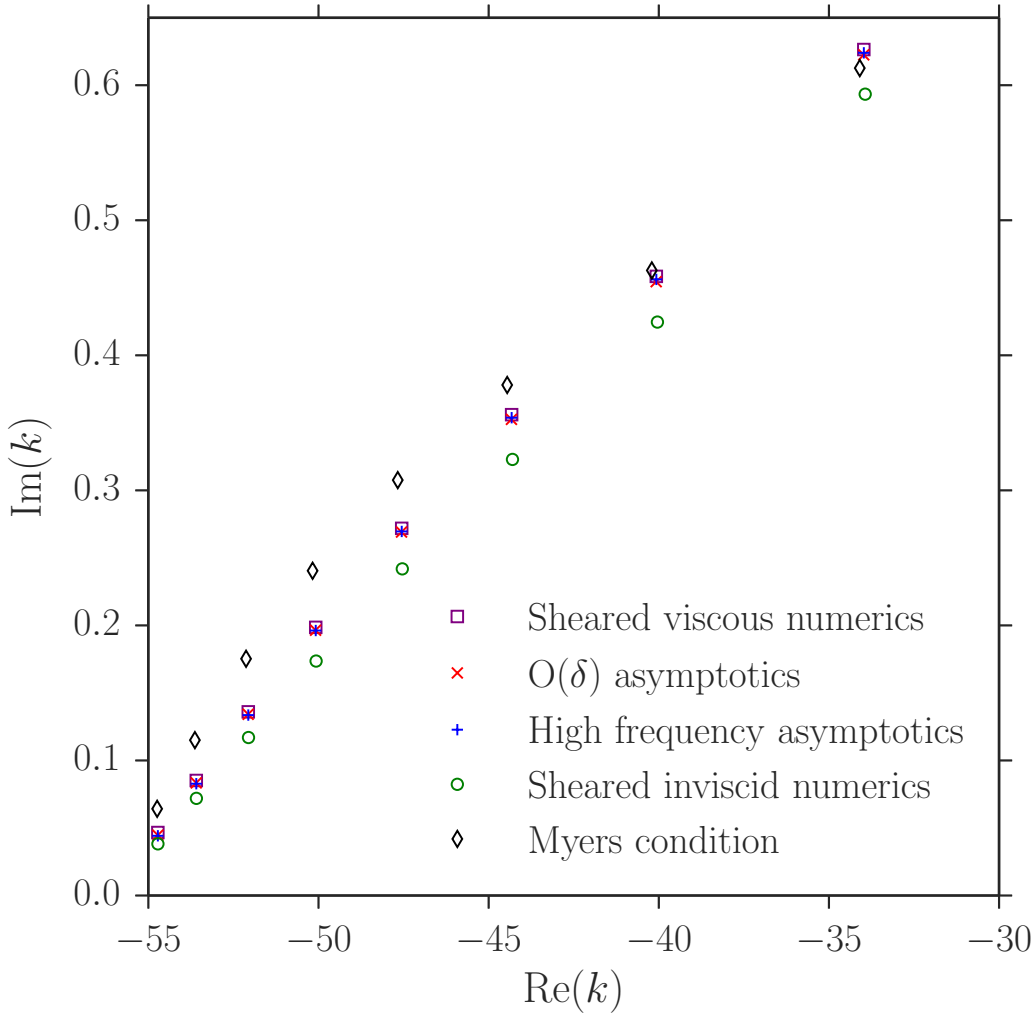
as $y \to \infty$ $\quad \hat{u}_1(y) \to u_{\mathrm{u}},\quad \hat{T}_1(y) \to T_{\mathrm{u}}.$

Effects of viscosity



- $M = 0.5$, $\delta = 5 \times 10^{-3}$, $m = 0$, $Z = 3 + \mathrm{i}(0.15\omega - 1.15/\omega)$, tanh boundary layer.

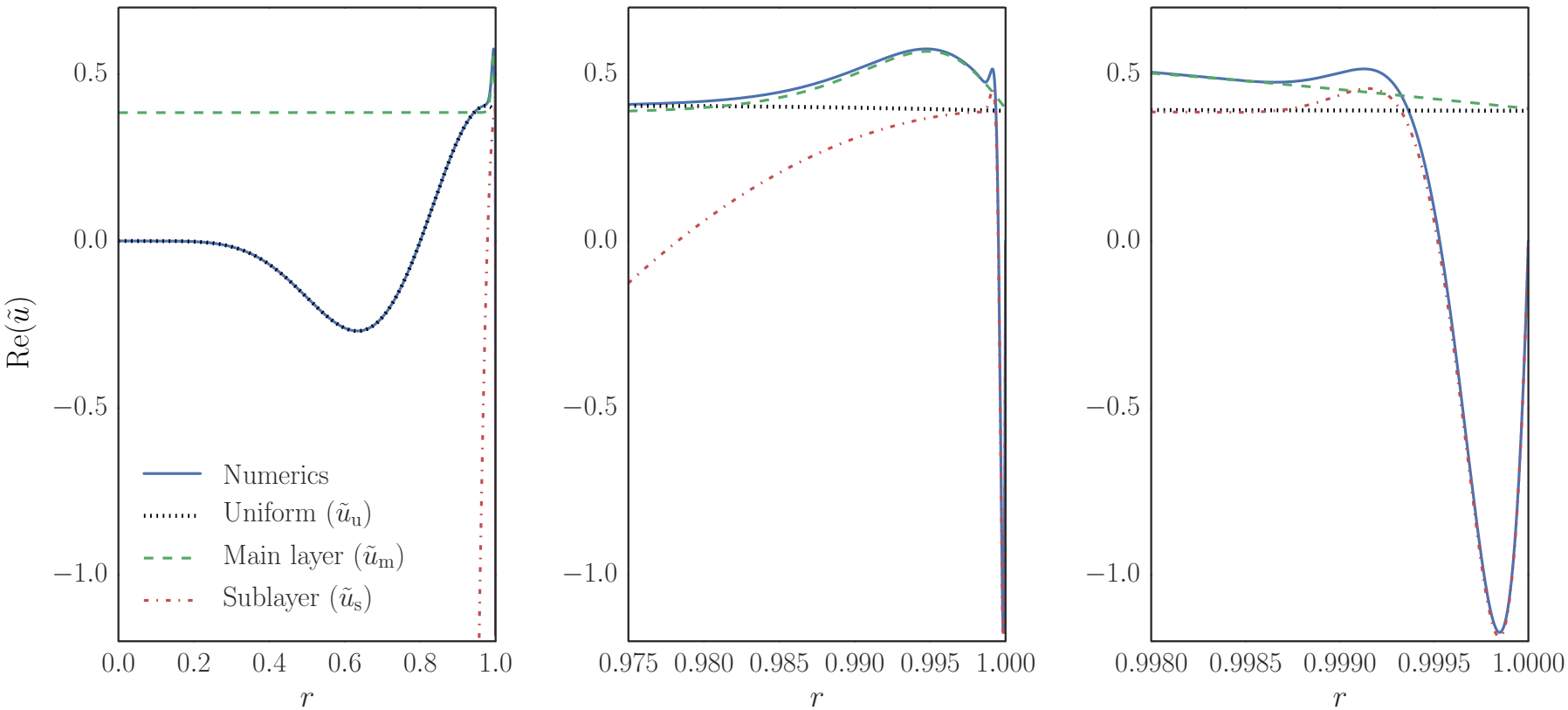
Effects of viscosity



- $\omega = 28$, $M = 0.5$, $\delta = 2 \times 10^{-3}$, $\mathrm{Re} = 5 \times 10^{6}$, $m = 0$, $Z = 3 + \mathrm{i}(0.15\omega - 1.15/\omega)$, tanh boundary layer.

Weakly viscous model (Khamis & Brambley, AIAA J 2017)

- Inspired by a turbulent boundary layer ($\delta \gg 1/\sqrt{\mathrm{Re}}$), we try $\delta \sim \mathrm{Re}^{-1/3}$.



$\omega = 15, \quad k = 5 + 2ß, \quad m = 6, \quad M = 0.5, \quad \delta = 7 \times 10^{-3}, \quad \mathrm{Re} = 3 \times 10^{6}.$

Axial velocity

Effective impedance

$$Z_{\text{eff}} = \frac{\omega}{\omega - Mk} \frac{Z + \frac{(\gamma-1)T(1)}{\sqrt{\mathrm{i}\omega\mathrm{Re}}} \frac{kU_r(1)}{\omega} Z - \frac{\mathrm{i}}{\omega}(\omega - Mk)^2 \delta I_0 + (S_2 + S_3)Z}{1 + \mathrm{i}(k^2 + m^2) \frac{\omega Z}{(\omega - Mk)^2} \delta I_1 + S_1 Z} + \mathcal{O}(\delta^2),$$

where

$$S_1 = \frac{(\gamma-1)T(1)}{\sqrt{\mathrm{i}\omega\mathrm{Re}}} \left(\frac{k^2 + m^2}{(\omega - Mk)^2} \mathrm{i}kU_r(1)\delta I_1 + \frac{\mathrm{i}\omega}{\sigma}(\gamma - 1) + \frac{\mathrm{i}}{\omega\rho(1)}(k^2 + m^2) \right),$$

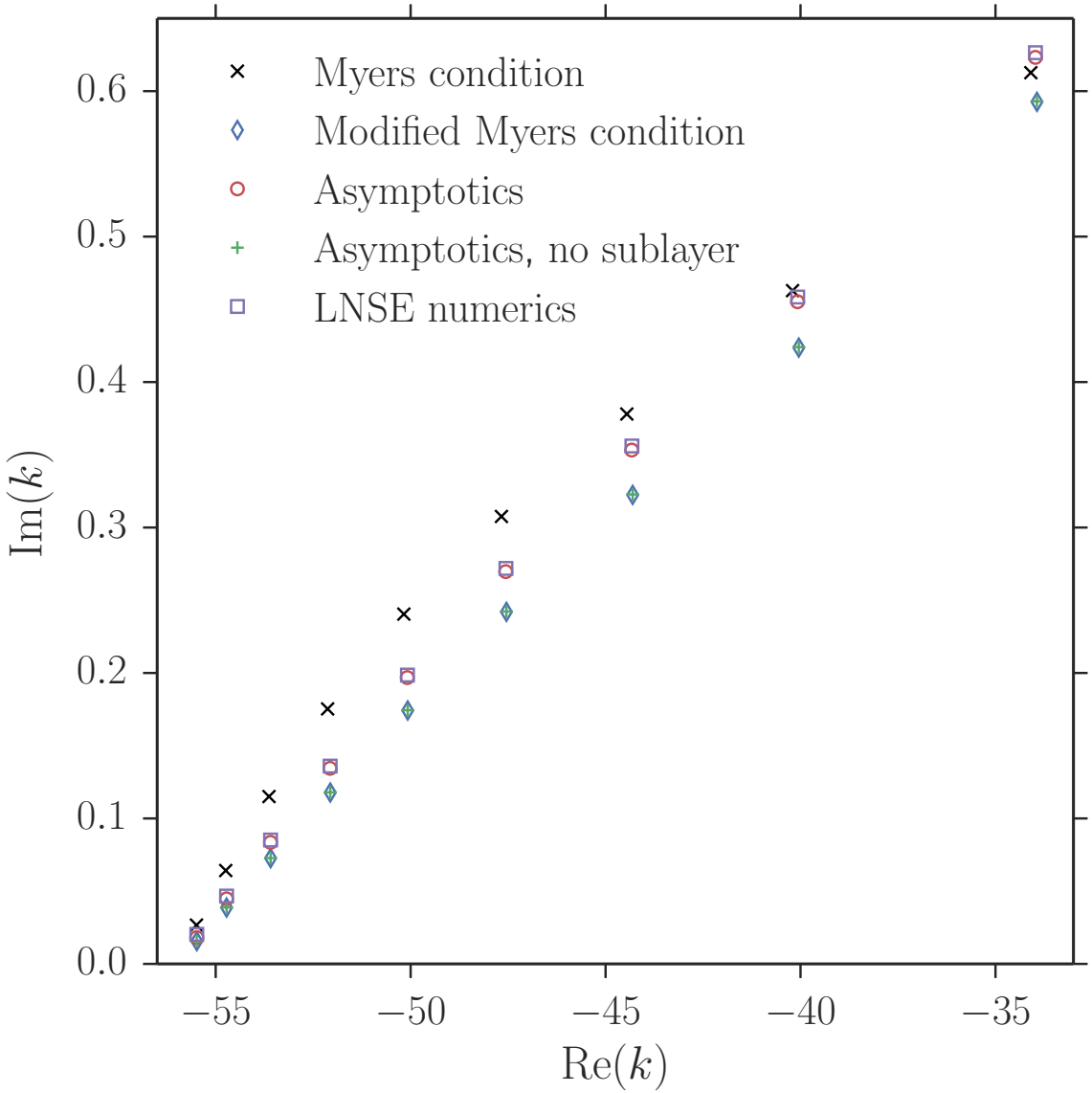
$$S_2 = \left(\frac{(\gamma-1)T(1)}{\sqrt{\mathrm{i}\omega\mathrm{Re}}} \right)^2 \left(\frac{1}{T(1)^2} \frac{I_\mu}{\delta^2} + \frac{\sigma}{1+\sigma} \frac{2U_r(1)^2}{T(1)} - \frac{5k^2}{4\omega^2} U_r(1)^2 \right),$$

$$S_3 = \left(\frac{(\gamma-1)T(1)}{\sqrt{\mathrm{i}\omega\mathrm{Re}}} \right)^3 \left(\frac{kU_r(1)}{\omega T(1)} \frac{I_\mu}{\delta^2} + \frac{13k^2}{8\omega^2} U_r(1)U_{rr}(1) + \frac{k}{\omega} U_{rrr}(1) + \frac{T_{rrr}(1)}{\sigma^3 T(1)} \right.$$
$$+ \frac{151k^3}{32\omega^3} U_r(1)^3 - \frac{(7\sigma + 3)}{(1+\sigma)^2} \frac{kU_r(1)^3}{2\omega T(1)} - \frac{(\sigma^3 + \sigma^2 - 2\sigma - 1)}{\sigma(1+\sigma)^2} \frac{2U_r(1)T_{rr}(1)}{\omega T(1)}$$
$$\left. + \frac{(2\sigma^2 + 4\sigma + 1)}{(1+\sigma)^2} \frac{kU_r(1)T_{rr}(1)}{\omega T(1)} \right),$$

$$\delta I_0 = \int_0^1 1 - \frac{\rho(r)(\omega - U(r)k)^2}{(\omega - Mk)^2} \mathrm{d}r, \qquad \delta I_1 = \int_0^1 1 - \frac{(\omega - Mk)^2}{\rho(r)(\omega - U(r)k)^2} \mathrm{d}r.$$

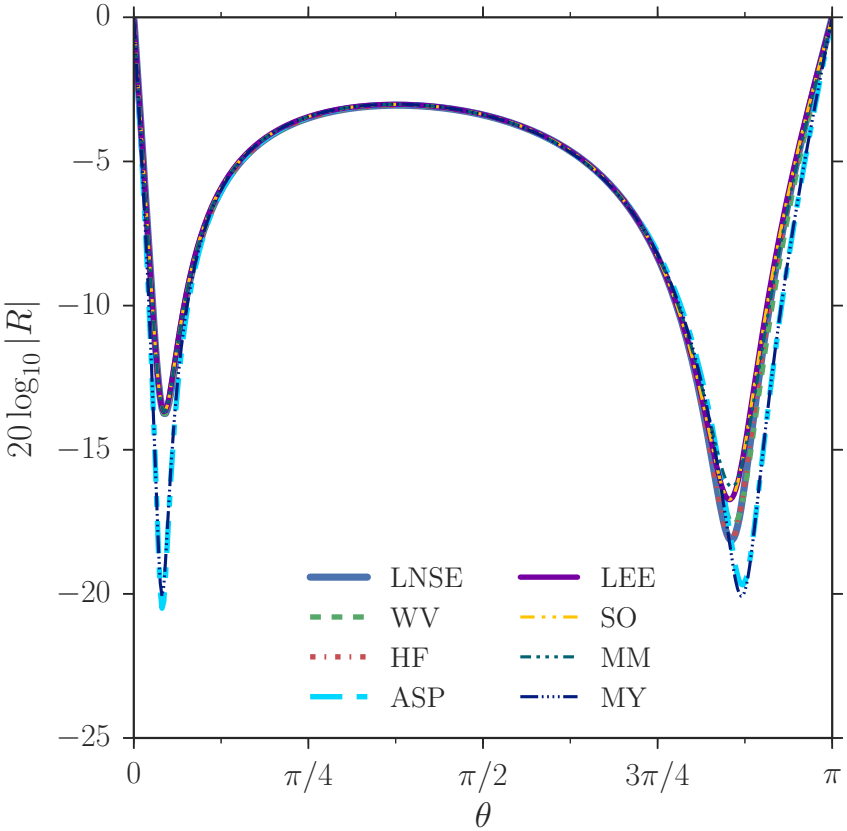
$$\frac{I_\mu}{\delta^2} = \int_0^1 \frac{-\omega}{\omega - Uk} \left(\frac{1}{2\mathrm{Pr}} (T^2)_{rrr} + (TU_r^2)_r + \frac{kT}{\omega - Uk} (U_r T)_{rr} \right) \mathrm{d}r$$

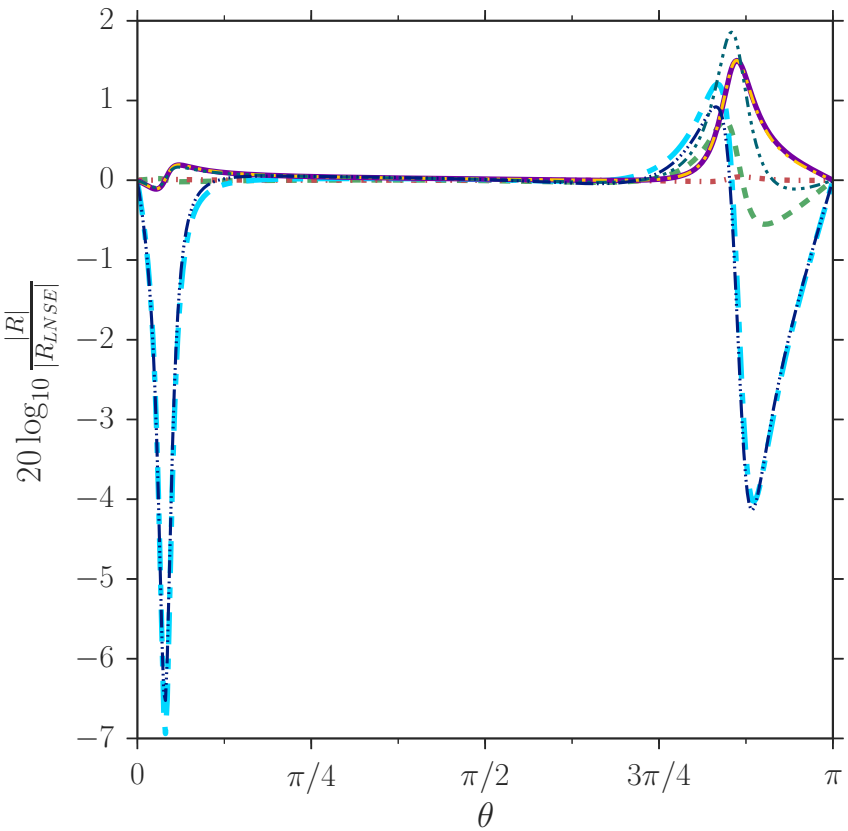
Effect on upstream modes



$\omega = 28$, $m = 0$, $M = 0.5$, $\delta = 2 \times 10^{-3}$, $\mathrm{Re} = 5 \times 10^{6}$.

Plane wave reflection (Khamis & Brambley, JASA 2017)





LNSE = Linearised Navier–Stokes equations, WV = weakly viscous two-deck,
HF = high frequency viscous, ASP = Aurégan, Starobinski & Pagneux,
LEE = Linearised Euler equations (LEE), SO = Second Order Inviscid,
MM = first order inviscid, MY = Myers.

$\mathrm{Re} = 5 \times 10^6$, $\omega = 28$, $\delta = 1.4 \times 10^{-2}$, $M = 0.55$, $Z = 5 - \mathrm{i}$.

Other models

- Non-asymptotic approximate solutions across the boundary layer (as yet all invsicid)
  (Ko, JSV 1972; Färm, Boij, Glav & Dazel, AA 2016; Khamis & Brambley, JFM 2016)

- Weak nonlinearity in the boundary layer
  (Petrie & Brambley, AIAA 2017)

- Swirling flow
  (Masson, Mathews, Moreau, Posson & Brambley, JFM 2017)

- Time domain formulation and numerical implementation
  (Brambley & Gabard, JCP 2016)

Still needed...

- Simple time-domain boundary condition, which is:
  - simple (no convolution over space or past history);
  - accurate (correct impedance eduction upstream and downstream); and
  - has the correct stability.

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Thank you for your attention!